THE
MATHEMATICAL GAZETTE.

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PRESIDENTIAL ADDRESS.

BY SIR G. GREENHILL.

MATHEMATICS IN ARTILLERY SCIENCE.

"THERE is no such thing," the artillery officer would have answered less than six months ago. And you were not to tease him with the name Scientific; the obliquity of the compliment, he explained, was liable to do him harm in the esteem of his Seniors.

But this outlook is Ancient History, and at the present time the relation of Mathematics to Military Science is engaging the deep attention of us all.

This is a Mathematical War, and so I propose to offer some reminiscences of my experience as a Professor of Artillery Theory, and to cite a few instances where Science can prove itself on Service useful and honourable.

To keep under the half-hour limit, a selection must be made of illustrative applications of Theory, say to Geometry, Dynamics, Sound, and Conduction of Heat.

These illustrations must be chosen on a simple, humble scale, so as to be comprehended elsewhere, trivial as they may seem to this

learned company, so begin with an application of Euclid's Geometry; to take the fragment of the wall of a shell, and determine the calibre, a simple application of Proposition XXV, Book III; not Proposition I, which requires the complete circle to be drawn. We can tell then to a certainty whether it came from one of the fabulous 42 cm. (17-inch) howitzers, the existence of which still appears in doubt, or from the 28 cm. (11-inch), quite small by comparison.

For a Dynamical application we look at the pictures in the illustrated papers, *Engineer* or *Sphere*, purporting to be from photographs of these weapons, and determine the scale and weight, not merely from the men around, but an inspection of the railway track, and the axle load permissible, making allowance for gross, tare, and net weight. The length of recoil allowed by the hydraulic buffer will settle an estimate of the muzzle velocity, and the calibre gives the weight of the shell as readily as counting the courses in a brick wall will determine the length required of a scaling ladder.

Sound Theory, bearing on Deafness; the propagation of Sound is a more humane subject, of medical interest; it teaches us how near it is safe to stand to a gun when it is fired, without danger of permanent deafness. I noticed in my time so many artillery officers had lost an ear in action, always, I was assured, from standing too near a small gun or mortar.

A working rule to employ is to make the minimum safe distance to vary as the cube root of the charge, or say, as the calibre or cube root of the weight of the shot.

Taking as the constant of calculation the old 64-pounder converted gun, formerly the arm of the Artillery Volunteer, when he was instructed to stand about 4 yards behind the muzzle, he ought not to advance closer than one-quarter of the distance, or 1 yard behind the muzzle of the one-pounder. And going to a larger size, say from 6-inch calibre to 18 inches, he need not fear to stand 12 yards behind the 42-centimetre howitzer; and so the story is discounted of the firing-party taking cover 100 to 200 metres away when this howitzer is fired.

The rule assumes that in the explosion of a spherical charge of powder of unit density, ignited and converted into gas at a normal pressure, say of 20 tons/inch^2, a spherical sound wave is thrown off, in which the displacement in the air and its physical effect on the ear or a window is inversely as the distance, but the energy inversely as the distance squared.

It is employed in the regulation of the storage of explosives near a town, and the necessary constant was supplied by the blowing up of the magazine in Erith marshes fifty years ago, which smashed the windows of Woolwich.

Gunpowder makers complain that the Government rule is unduly severe; but they only prove that it was likely the amount of powder in the Erith magazine was much greater than the owner would confess. The rector of Plumstead was encouraged thereby, however, to calculate that his church tower was being shaken down by the firing of the big Woolwich guns at proof, and to make a demand on the War Office for rebuilding of his church.

In Conduction of Heat an application of the theory would have reassured our men that Life in the Trenches would not be too cold; or at least warmer than in the frost above, provided only the floor can be drained dry under foot; the trench is better cover than a tent. In the linear vertical conduction of heat down through the ground, due to an annual (T) climatic variation of temperature (V), given by

$$(1)\quad V = A + B \sin nt,\quad n = \frac{2\pi}{T},\ \text{in the air,}$$

the equations of conduction of heat v,

$$(2)\quad k\frac{d^2v}{dx^2} = c\frac{dv}{dt},\ \text{in the ground;}$$

$$(3)\quad k\frac{dv}{dx} = h(v - V),\ \text{at the surface, } x = 0;$$

are satisfied by

$$(4)\quad v = A + v_0 e^{-px}\sin(nt - \epsilon - px);$$

$$(5)\quad 2kp^2 = nc,\quad \cos\epsilon = \left(1 + \frac{kp}{h}\right)\frac{v_0}{B},\quad \sin\epsilon = \frac{kp}{h}\frac{v_0}{B};$$

so that the heat is propagated downward in a damped wave of length

$$\frac{2\pi}{p} = 2\pi\sqrt{\frac{2k}{nc}} = 2\sqrt{\frac{\pi kT}{c}},$$

travelling with velocity $\dfrac{n}{p} = 2\sqrt{\dfrac{\pi k}{T}},$

and damped with a naperian log. decrement 2π.

The phase lag at the surface is ϵ, equivalent to a time lag

$$\frac{\epsilon}{2\pi} = \frac{1}{2\pi}\tan^{-1}\frac{kp}{h + kp}\ \text{of the year.}$$

Also $\dfrac{v_0}{B} = \cos\epsilon - \sin\epsilon$; and the frost will reach a depth x, where

$$v_0 e^{-px} = A,\quad e^{px} = \frac{v_0}{A} = \frac{B}{A}(\cos\epsilon - \sin\epsilon),$$

$$x = \sqrt{\frac{2k}{nc}}\log\frac{B}{A}(\cos\epsilon - \sin\epsilon).$$

The formula is used to calculate the depth to bury a water main to escape being frozen. We realise the theory with the water from a well, or in the Tube Railway, where the climate is so equable as to appear warm in winter and cold in summer, suitable for a consumptive cure, except for absence of daylight.

But School Board Science teaches the children never to take service in a house with a basement; with the result that in the spreading of new London the villas are built flat on the ground, cold and rheumatic for mistress and maid. Old-fashioned houses in St. John's Wood cannot let to-day, as it is impossible to get a servant to live in them, the house being built on the London clay, and so provided with a basement dug out.

Five years ago I had an invitation to Berlin to visit the Military Technical Academy there. Here is a magnificent institution, such as we couldn't afford, our rulers thought, and assured us. Imagine some of the new buildings at Kensington, with the same collateral advantage of being next door to the Charlottenburg Central College, available for lectures and laboratories of pure peaceful Science; and the whole place kept smart and noble as one of our old colleges, a great contrast to our shabby military buildings.

Professor Cranz showed how in his department no money was spared in every recent equipment, including a bomb-proof range available for artillery fire, and yet in the heart of a big city. Plenty of outdoor artillery ranges too, close at hand to visit when instructive work was in progress.

The Perry system of education was adopted in Berlin. After a lecture on Wireless Telegraphy, the class was set to work as I saw in making the antennæ, which have played such an important part in the war. But it was Squeers, long before Perry, invented this system of combination of Theory and Practice, only he was ruined by the incompetence of his demonstrator, Nicholas Nickleby, the young hero of the story in the author's opinion, and of old-fashioned schoolmasters still.

Sixty officers are under instruction there at a time for a course of three years, and I was assured their zeal is admirable. It was considered such bad form not to give the best in return for the Honour and Glory of the Fatherland. But our Regular was apathetic by comparison; or, as he would say of himself, Not keen. We must put our trust in the junior ranks to carry us through this war. But first push old Apathy from his stool.

It was a mournful contrast to revert to Woolwich, shabby and undisciplined. There we had been evicted from our proper home, and were told to found a new Artillery College, with the choice of a cellar under some stables, or a kitchen and scullery and bare walls in a deserted hospital, there to organise Victory, and at no expense.

With the courage of an Austrian general, compelled to maintain his muzzle-loading musket a match for the Prussian needle gun, our Military Director assured us there was nothing superior to be found even at Greenwich, in the Naval College there, lodged in the old Palace.

Such dismal penurious surroundings had a disastrous effect on the *genius loci*, and we never really recovered from a down-hearted spirit, not calculated for Victory.

Our Military Science is under the Rule of Thumb, the official genius. His fumbling method is considered a match for disciplined Theory. And Thumb is hostile to Theory. He scents a dangerous rival, to be swept from his path ; under a delusion he can stand up to disciplined Theoretical Science.

We see already how the cost has been well laid out in this war of the Berlin Military Technical Academy, the German jumping off with a lead, and able to keep it so far. The finished article of the Academy is employed in the dissemination of true Theory, and in the scientific direction of warlike preparation, as at Krupp's. Thumb and his fumbling were thrown on the scrap heap in Germany, as too expensive and inefficient.

Assuming everything for the best for the Allies, and if we live to go in again at Antwerp, to knock up the pistol at our head and keep our word to Belgium, an interesting match will be watched between our Artillery Science and the German, to see how long it will take us to get the other side out, compared with our own innings, and the time we kept our wicket up.

No Long-Range Fire, I was always assured, was ever going to be of any use again, involving theoretical calculation.

The word was "gallop up close, to 400 yards, and let them have it," in the primitive artillery tactics of Colenso. Ballistic Theory, more primitive still, and at a range still shorter, it was not realised then would come in useful in the trenches, in throwing the primitive Hand Grenade.

The country is furious at the way our poor fellows were pounded mercilessly at the start, by long-range, accurate howitzer fire of the German Artillery, with no protection from our own side.

King George's stirring appeal,

WAKE UP ENGLAND,

was intercepted by our Rulers, and it was England the Unready again, when our Senior Lethargy came bump into the Titanic Energy of the German Empire. G. GREENHILL.

The President called upon Dr. William P. Milne to read his Paper on:

THE TEACHING OF MODERN ANALYSIS IN SECONDARY SCHOOLS.

In considering pedagogic questions relating to mathematics, I think we are inclined to devote too much attention to discussing modes of presentation and class-room detail and not to pay sufficient attention to the political aspect of the point at issue,—if one may use the word "political" with reference to the mathematical world exactly as one does with regard to the realm of civil affairs. Certain reforms in teaching are brought to pass after much discussion and often much opposition, but we rarely discuss in our public meetings what ultimate forces operating in the mathematical world have brought about the need for these reforms. I propose to review now briefly the political state of Modern Analysis, with a view to suggesting some much-needed improvements.

The discovery of mathematical truths is carried out chiefly by a body of men of the requisite ability and originality who are provided with the necessary opportunity by the Universities of the various countries. These investigators, either through the medium of their lectures or books, communicate their discoveries to others who act as teachers. Also, the investigators themselves often take part in setting the papers at the various public examinations, for which the teachers are constantly engaged in preparing pupils. Thus, these discoverers, both consciously and unconsciously, fill the minds of their own pupils with the new ideas and, what is equally important, the new trend of ideas as the subject unfolds itself under their own and others' labours. Furthermore, their examination papers reflect the results of the discoveries being made by indicating the trend which the subject has taken and is likely to take. We may, therefore, conclude that, broadly speaking, the active researchers ultimately and in the long run—though often not immediately—fix the nature of the teaching that is given in our schools and colleges.

Let us now apply these principles to the teaching of Modern Analysis in Secondary Schools. If we consult the memoirs published in this country—for example the *Proceedings of the London* or *Edinburgh Mathematical Societies*—we find that a large proportion of their pages is devoted to researches on Modern Analysis,—that is, Limits, Series, Approximations, and the like. We find that attention to these subjects is forced upon them primarily by the requirements of Astronomy, Tides, Chemistry, Motion of Bodies, and Flow of Heat and Electricity, and so forth; and secondarily, the desire to obtain the general criteria and properties common to all the particular functions that thus present themselves. Is it a wonder then that these subjects should be receiving ever-increasing attention in schools and universities?

We proceed next to consider the actual state of teaching that prevails with regard to these subjects. What we are going to say refers by no means to *all* the teaching throughout the country. Many able and enthusiastic teachers have emancipated themselves from the bonds of tradition, but I think that every earnest and reflecting mathematician is bound to admit that the instruction in Modern Analysis regarded as a whole is far from satisfactory in those of our Secondary Schools which are able to teach really advanced work.

The teaching of Modern Analysis proceeds in a Secondary School somewhat on the following lines: The teacher has enjoyed doing the Binomial Theorem for a positive integral index; he has felt perfectly

happy in carrying his pupil through the fascinating puzzles belonging to Permutations and Combinations; he has managed to evoke a good deal of interest in the Theory of Numbers and Summation of Finite Series. He may even teach Determinants at this stage to stave off for a little while longer the evil day which he feels all the while is advancing with slow but irresistible advance. At last it dawns, and he faces his class with a silent tirling at the heart when he says, "Boys, Limits and Convergence of Series must now be done." He starts with putting the statement $S_n \equiv u_1 + u_2 + \ldots + u_n$ on the blackboard and saying "that S_n is called convergent if S_n always remains finite however big n may be"—a definition which is perfectly inadequate. He puts down four or five tabulated tests which look as though they had dropped from the sky. He works out one or two examples to show how these can be applied to absolutely dreadful series—the like of which the pupil has never seen before and—be it said for his comfort—which he is never likely to see again. The boy is then set to work through several dreary pages till he can apply these tests almost literally with his eyes shut, and then the subject is considered as finished with and mastered. Now, what has the boy learned from this course of Modern Analysis? I think that we are bound to admit, "Practically Nothing."

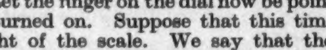
Let us now carry this investigation one stage further and review our own learning of Fourier's Theorem. One or, at most, two lectures were devoted to the study of Fourier's Theorem, and these lectures were mainly taken up either with a perfectly dreadful proof or with some dull and on the whole unilluminating illustrations. At all events we can only judge by the effect on ourselves. I think I can say that very few of us ever got at the soul of Fourier's Theorem. Personally I "got at" the essence of Fourier's Theorem not from any formal lectures or from any formal text-book which I had ever read, but from some hours two summers ago at the Edinburgh Mathematical Colloquium, where, provided with Analysing Paper, Multiplication Tables, and Calculating Machines, we actually analysed the periodic motion of a Star from the given recorded numerical data as taken down in one of the great observatories.

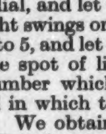
What is wrong then with our teaching of Modern Analysis? We can sum it up in five brief words, "*One essential stage is omitted.*" We do not start with the concrete, and build the abstract theory thereon. We begin with the abstract and do no concrete, and there is no good introducing our pupils to an abstract formal proof of a theorem which they have never used in practice. They must get their teeth into the subject first of all. They must themselves deal with concrete numerical cases until they are in a position to have satisfied themselves as to what it is all about. The concrete details must be in their heads, and they must have something to generalise from. If one is content to begin one's study of a subject by reading about it in an abstract and generalised form, one is taking merely a spectatorial interest therein. The ability to deal with and extract scientific truths from a vast mass of collected observational data is essential to a subsequent profitable study of general Function Theory.

Let us now proceed to some of the practical details of teaching Modern Analysis. It will be impossible to give the pupil straight away a clear idea of the pitfalls and difficulties that beset the fundamentals of the subject. In learning any branch of Science, one has got to risk a good deal at the start. One must first of all capture new ground and then return to strengthen and fortify it. A celebrated engineer has said, "I treat all my series as convergent till a boiler bursts, and then I return to see which of my series were divergent." This is what we all do when dipping into a new subject, and we must

perforce allow our pupils to do the same. Hence we cannot present to our pupils the fundamentals of Modern Analysis in the way that mathematical philosophers regard such fundamentals. We must dovetail the beginnings of Modern Analysis as far as possible into the previous arithmetical and algebraic work of the school. After much class-room experimentation I have found that the Dedekind-Schnitt method of defining an Irrational Number is unnatural and hard for the pupils to grasp. The method of Infinite Decimals is pedagogically superior and fits on to their previous work better. They already know that $\frac{1}{3} = \cdot 33333...$ to ∞, that $\frac{2}{7} = \cdot 285714285714...$ to ∞, that $\sqrt{2} = 1 \cdot 4142...$ to ∞, and so on. Hence the statement that we will regard every irrational number as defined by a non-terminating decimal is no utterly new conception. On the other hand, this method of definition has the disadvantage that the pupil is apt for a long time to regard an irrational number as not being an exact number but of the nature of an approximate entity. No amount of "talk" will dispel this just at first, but graphical work showing the successive stages by which 1, 1·4, 1·41, 1·414, 1·4142, ... approximates to the marked point $\sqrt{2}$, and many other such illustrations will show the pupil exactly where he is and give him an accurate idea of what the definition of an irrational number by a non-terminating decimal really involves. Furthermore, we can show that this definition is the *modus operandi* of the Physics Laboratory. In Sir Oliver Lodge's Presidential Speech two years ago before the British Association, he pointed out, in what to the schoolboy is a more convincing way than I have ever seen elsewhere, the importance of the "irrational number" in physical measurements. He pointed out that, if we take a ruler subdivided in the usual way, however minutely, most of the lengths to be measured fall not exactly *on* but mostly *between* the subdivisions —whether they be tenths or sevenths or elevenths, etc. In fact, if we can conceive measurements capable of being carried out to an ideal state of accuracy, most of the lengths presenting themselves in Nature will be found to be capable of representation only by "irrational numbers." So it is with weighing. Again, in the Laboratory the pupil will be very familiar with the following process :

Scale of Frosted Glass.





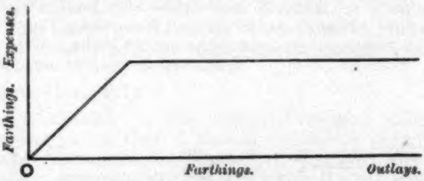
A spot of light from a swinging mirror is allowed to move from one end to the other of a scale made of frosted glass. Suppose that the finger on the dial is pointed to 3 and that the electric current is turned on, and suppose further that the spot of light swings to the left of the scale. Let the finger now be pointed to 4 on the dial, and let the current again be turned on. Suppose that the spot of light swings once more to the left. Let the finger on the dial now be pointed to 5, and let the current again be turned on. Suppose that this time the spot of light swings to the right of the scale. We say that the number which we want lies between 4 and 5. We then use another dial in which the interval between 4 and 5 is divided into *10* equal parts. We obtain as before that the number we want lies between (say) 4·7 and 4·8. So on we proceed, obtaining a closer and closer approximation as far as the instruments at our disposal will allow. The pupil is now familiar with the definition of a number by means of a non-terminating decimal, and more than that, he finds that he has been using this definition unwittingly for longer than he thought. The important theorem can now be proved to him that *all* non-terminating decimals do not repre-

sent "irrational numbers," but that "recurring" decimals represent the realm of "rational numbers." Approximations and Limits of Error follow, and by the time that the pupil has worked out a large and varied assortment of numerical cases on these lines, he knows a good deal about the spirit and aims of the Theory of Functions,—which he never learns by doing complicated "Convergence Tests."

I must next say a word about "graphs." Graphical work is of the utmost importance in this branch of mathematical work, but as usually taught the jump is too sudden from the engineering "trust-in-Providence" definition of a graph, to the more rigorous definition required by Modern Analysis. Similarly the abstract definition of a "function" is thrust too soon and too suddenly on the boy. The symbolism $y = f(x)$ really means that the user has in his head a vast number of concrete numerical examples and properties of various functions, and that $y = f(x)$ is a sort of mental abstraction from all these. In beginning the study of the Theory of Functions, the vagaries of $y = \sin \frac{1}{x}$ at the origin should be ruled out. It is really very hard, and the pupil cannot see all the cranks and wheels working as he proceeds. On the other hand, $y = I(x)$ (where $I(x)$ means the integer next less than x) is very good, as the pupil sees exactly what he is doing. I will give three examples of functionality which can be very profitably presented to beginners in the Theory of Functions, and the oral discussion of which will go a long way towards showing them what the subject means.

EXAMPLE I. *An external examiner in an educational institution is allowed his travelling expenses, not exceeding four pounds. Exhibit this graphically.*

Obviously the graph is of the following form:



The outlay and expenses are equal till the sum of £4 is reached—after which the sum of £4 is paid, whatever be the outlay. Also it ought to be pointed out that whereas in Engineering and Practical applications the above would be represented by straight-lines, in Modern Analysis the graph is really a series of collinear dots, inasmuch as nothing smaller than a farthing can be used.

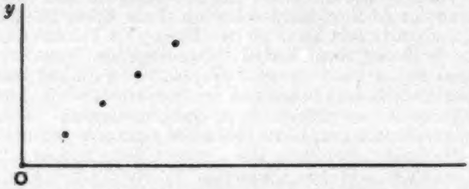
EXAMPLE II. *Draw the graph of $x = I(t)$, $y = I(t)$, where I denotes the integer next less than t.*

This graph is useful in showing the pupil from his own experience that every graph is not continuous in the geometrical sense and need not have a "length," even though the independent variable t take all values rational and irrational from 0 to $+\infty$.

Consider the following table:

t	0	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{3}{4}$	1	$1\frac{1}{3}$	$1\frac{1}{2}$	$1\frac{3}{4}$	2	$2\frac{1}{3}$	$2\frac{1}{2}$	$2\frac{3}{4}$	3	π
$I(t)$ x y }	0	0	0	0	1	1	1	1	2	2	2	2	3	3

The graph is therefore a series of collinear equidistant dots as shown in the figure.

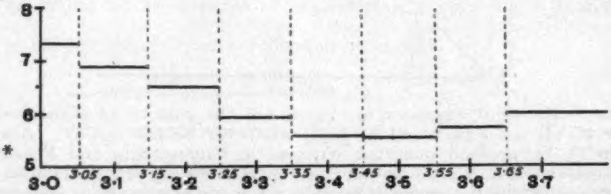


Example III. A very good example can be obtained from the ordinary everyday use of sets of numerical tables.

Suppose we look up a set of tables and obtain the following correspondences:

x	3·0	3·1	3·2	3·3	3·4	3·5	3·6	3·7	3·8	3·9	4·0
y	7·3	6·9	6·5	5 9	5·6	5·5	5·7	6·0	6·7	7·2	7·9

In ordinary practical or engineering work we should plot the above points on squared paper and "sketch in" a smooth graph. The exigencies of Modern Analysis require absolute accuracy however, and we must examine more closely how we deal in practice with the above tabulated data. When x has any of the values 3·0, 3·1, 3·2, ... 4·0, we take the corresponding value of y directly from the above table. Suppose that to x is assigned any other value such as π; we take π correct to the first place of decimals and then "read off" directly from the above table. Hence the graph of the above table as actually used in practice is



the usual convention that 3·25 is to be called 3·3 being maintained.

Now, one usual "graph" in elementary books on Analysis is "*$y=0$ if x be a rational number and $y=1$ if x be an irrational number.*" This function does not appeal to the student, as it seems so hopelessly artificial. In the above graph, however, he obtains a case taken from actual everyday practice, where the values assigned to the independent variable x have to be treated differently in obtaining corresponding values for y according as they are one or other of the tabulated values 3·0, 3·1, 3·2, etc., or intermediate values.

After the pupil has gone through a considerable numerical course, somewhat on the above lines, he will be able to think quite accurately of y depending functionally on x, and he will have a considerable stock of concrete varied phenomena in his mind, so that he will be able to

* In the above diagram the right-hand ends of the black horizontal lines approach indefinitely close to the dotted vertical lines, but never actually reach them.

appreciate some of the seemingly useless and artificial definitions and restrictions that usually appear to be flung arbitrarily at his head in the opening pages of treatises on the Theory of Functions.

With regard to what can be taught successfully to Scholarship candidates, there must be applied a process of elimination to the existing schedule such as has been applied so successfully in the Secondary Schools and Technical Colleges. If much of the existing useless matter is cut out of our schedule of Higher Algebra, we will have gained the necessary leisure for the successful teaching of a fuller and wider scheme of Modern Analysis along the lines and in the direction indicated by the researches of the great masters. I suggest, therefore, that a scheme of Modern Analysis beginning with the elements numerically treated, containing a numerical discussion of approximations and limits, dealing numerically with polynominals and power-series, and concluding with the differentiation and integration of power-series, would form a very suitable scheme for inclusion in our schedule of Scholarship Work.

Probably most progress would be obtained by a definite break with the past, and hence I suggest that Modern Analysis be separated from Higher Algebra properly so called—Higher Algebra to contain all the usual work not involving Limits; *e.g.* Summation of Finite Series, Theory of Integral Numbers, Determinants, Permutations and Combinations, etc., and Modern Analysis to be devoted to Limits and kindred subjects. In this way teachers would face the new aspect with fresh minds, and would to a large extent get rid of the numbing effect of tradition (for we are all by nature Conservatives).

I cannot conclude this paper without reference to the admirable treatise on *The Teaching of Algebra* just produced by Prof. T. P. Nunn. He has attacked many of the above problems, and without pretending to give final conclusions, he has shed a flood of light on the whole pedagogy of Irrational Numbers and kindred subjects, and has given us one of the most important pedagogical books on Mathematics that has been produced for some time.

Added February 12th, 1915:

During the discussion on the paper, Professor Alfred Lodge, of Charterhouse, suggested that I should outline a definite scheme of work on Modern Analysis for Secondary Schools, assuming that it were treated as a separate subject from the Higher Algebra of Non-Limit subjects. It is not possible *as yet* to frame what could in any way be regarded as a perfectly satisfactory working scheme, as there is at the present moment so little pedagogic experience that can be brought to bear under this heading. The following is, however, a perfectly practicable scheme, and one that has to some extent borne the brunt of experience:

I. Much practice on numerical inequalities (*e.g.* if $l=7{\cdot}924$ and $m=8{\cdot}432$, each correct to three places, to how many places do we know $l+m$, $l-m$, $l\times m$, $l\div m$?).

II. Numerical sequences and their properties (*e.g.* find the points on $x'ox$, such that, for all values of x lying between them, $x-5x^2+7x^3<{\cdot}01$).

III. Introduction of the rigorous conception of a limit as defined by an infinite decimal. The study of infinite decimals which define rational numbers and those which define irrational numbers. Much numerical work on infinite decimals and their limits.

IV. Some easy properties of sequences (*e.g.* if $s_1, s_2, s_3, \ldots s_n$ steadily increase but never become greater than K, then the sequence approximates to a limit $s \leqslant K$). Much numerical work to drive these ideas home.

V. Properties of Polynomials—differentiation and integration.

VI. Easy Series.

VII. Complex Numbers.

VIII. Study of Polynomials by means of the Argand Diagram.

IX. Study of $\dfrac{a_0z^m+a_1z^{m-1}+\ldots+a_m}{b_0z^n+b_1z^{n-1}+\ldots+b_n}$ – roots, poles, expansions at points, and residues. Much numerical work and generalisation of numerical properties that are always turning up. (This foreshadows practically the whole theory of Algebraic Functions, and opens up a vast field.)

X. Differential and Integration of Power-Series on the Argand Diagram.

XI. Only such Limit and Test-Theorems as are directly required by the above simple course. Nothing worse than the direct comparison with the Geometrical Series should be involved WILLIAM P. MILNE.

The President then requested Mr. R. C. Fawdry to read his Paper on:

LABORATORY WORK IN CONNECTION WITH MATHEMATICS.

ONE of the papers read last year at the Annual Meeting of this Association was on the subject of Practical Mathematics, but much of the value of the paper was lost as time did not permit of any discussion. To-day I propose to make my remarks very brief, as my chief object is to ascertain what steps are being taken in schools of various types, to bring the subject of Practical Mathematics into the School Course. I trust, therefore, that there will be an expression of opinion from as many members as possible.

In order to avoid any misunderstanding, I have entitled my paper "Laboratory Work in connection with Mathematics." This indeed is what I understand by Practical Mathematics. It will be obvious that such a meaning is widely different from that adopted by Professor Steggall in the remarks he made last year.

Numerical evaluation of algebraical expressions, accurate constructions of geometrical problems, plotting of curves, graphical solutions, use of logarithms in computation, in fact the bulk of the methods which have been adopted in the class teaching of Mathematics largely as the result of the efforts of the Mathematical Association—these to me do not mean Practical Mathematics. Such operations can be conducted in a class-room without the use of further apparatus than a box of instruments, some squared paper, and a table of logarithms.

Those who are familiar with the examinations conducted by the Civil Service Commission will be aware that the examination in Mathematics includes a practical test, conducted according to a prescribed syllabus, and it is in great measure due to the demands of these examinations that the attention of teachers has been drawn to the advisability of including practical work in the Mathematical Course.

Practical Mathematics according to this view requires something in the nature of a Laboratory. Broadly speaking, it is the application of Mathematical processes to data which have been obtained by the pupil as the results of experiments he has performed.

One obvious result of such practical work is the infusion of life into the dry bones of the technical procedure with which youthful mathematicians must be familiar before any great progress can be made in a Mathematical education.

Professor Whitehead, in his little book *An Introduction to Mathematics*, states :

"The study of Mathematics is apt to commence in disappointment The important applications of the science, the theoretical interest of its ideas and the logical rigour of its methods, all generate the expectation of a speedy introduction to processes of interest. We are told that by its aid the stars are weighed and the billions of molecules in a drop of water are counted. Yet, like the ghost of Hamlet's father, this great science eludes the efforts of our mental weapons to grasp it, and what we do see does not suggest the same excuse for illusiveness as sufficed for the ghost, that it is too noble for our gross methods.

"The reason for this failure of the science to live up to its reputation is that its fundamental ideas are not explained to the student, disentangled from the technical procedure which has been invented to facilitate their exact presentation in particular instances."

We are making some attempt to prevent the study of Mathematics from beginning in an atmosphere of discontent and disappointment. The memory of our school days has made us merciful. The early steps in Arithmetic which plunged us into endless long division sums, the horrors of $5\frac{1}{2}$ yds. and $30\frac{1}{4}$ sq. yds., the interminable G.C.M.'s which wound their sinuous course about our pages—these, it is true, have largely disappeared, but the necessity for acquiring facility in computation remains. The introduction of practical work is to make these operations deal with data which are real; in short, we wish to stimulate interest.

Instead of giving the pupil a list of corresponding values of x and y and asking him to draw a graph to show the relation between them, we provide him with a stop-watch and a pendulum, and tell him to find the relation between the length of the pendulum and the time of an oscillation.

Instead of being told that if 56 sq. cms. of cardboard weigh 4·3 grams, find how many sq. cms. will weigh 10·2 grams, he is told to find the area of an irregularly shaped piece of tinfoil by cutting it out in cardboard and weighing.

The weakness of the text-book question lies in the fact that the necessary data for solving the problems are selected *for* the pupil, and not only the data but generally the method as well, for the questions to be solved according to a given model are usually grouped together.

In the Laboratory, the essential parts of the data have to be selected, irrelevant information has to be rejected,—the boy must use discretion, and his education correspondingly gains. He also learns to use what knowledge he already possesses. It is a common experience that a boy who is familiar with the operations of the Calculus, completely fails to apply his knowledge to a practical question. He will find in his text-book a volume V expressed as a function of x, the depth—he will find $\frac{dV}{dx}$, and will probably explain what it means. Put him in a Laboratory with a bucket and tell him to measure the depth of water in it, for increments of 1 litre in volume—plot the results on a graph and find from the graph the area of the cross section at various depths, and he will come and ask you how to do it.

I propose to deal but briefly with the question of ways and means, as it is this point which I hope will be discussed by those of my hearers whose experience is greater than mine.

The ideal of course is to have a Mathematical Laboratory, or at any rate a room devoted to this particular work. If such accommodation is available the expense of equipment is small, since the apparatus is all the better for being simple and not elaborate. I shall be glad if

those members who have such a Laboratory will tell us how they consider it may be used to the best advantage. Those who have no such accommodation must fall back on the Physical Laboratory, but this is not often available for the purpose at the hours when the mathematician wishes to use it. It is most desirable that the Laboratory work should be taken by the masters who teach the mathematical set, but it is not all mathematicians who can be trusted to be let loose in a Physical Laboratory.

The alternative is to assign a definite period each week for Laboratory work, but this method results in the practical work being to some extent independent of the work which is being done in the mathematical set. It is then necessary to arrange a course of Laboratory work, more or less elastic, to be carried out by each set.

Here I should like to state my disapproval of the use of certain books which are published for this purpose. They contain a series of experiments, each accompanied by directions to be followed and a tabulated scheme with blank spaces for the numerical results obtained.

To my mind such a proceeding destroys one of the most valuable parts of the training. The boy must be told how to carry out the experiment, but he must write his own account of what he has done. He must make his own table of results of observations. By so doing he will in time acquire the power of expressing himself concisely in intelligible English and in a scientific manner. My own method is to have the details of the separate experiments pasted on stout cards, which are handed to the boys. The account of the work is written in note books in pencil, and, when satisfactory both as to accuracy and intelligibility, is signed by the master before the boy proceeds to the next experiment. A list of boys and experiments is kept, on which a record is made of the experiments completed.

It may be useful if I give in conclusion a brief outline of suitable work.

Elementary Stage.—Calculations of areas and volumes (invaluable for driving home the importance of significant figures and the necessity of giving results to the degree of accuracy justified by the data).

Illustrations of proportion, such as finding the area of an irregular lamina; calculating the weight of wood required for making a corner shelf from a given board.

Finding the area of curved surface of a cone made of paper by calculating the area of the sector of the circle into which it is developed.

Drawing graphs, *e.g.* length of pendulum and time of swing; oscillation of spiral spring with various weights attached; bending of a lath for given weights.

It is important that the graph, when completed, should be *used* to find some result not included in the observations.

Principle of Archimedes if such work is not included in the Physics Course.

Second Stage.—Use and principle of verniers, slide calipers, screw-wire gauge.

Numerous *Statical experiments,* of which it may suffice to mention the calculation of the efficiency of machines and the introduction of problems in three dimensions.

Dynamical experiments with Fletcher's trolley. Space-time graphs, velocity-time graphs. Acceleration of a sphere rolling down a plane. Experiments with Atwood's machine.

Reaction of a jet of water on the bend of a pipe, as in Ashford's *Dynamics*, p. 185.

Dynamical experiments are of great value in clearing up ideas with regard to units. Nothing is more common in experimental work than the habit of taking $g=32$, when all observations are made in C.G.S units.

More Advanced Stage.—Experiments involving the application of the Calculus.

Area of cross section of a vessel from observations of volume and depth.

Moments of Inertia. Oscillating lamina. Compound pendulum.

Kinetic energy of fly wheel.

Applications of coordinate geometry in three dimensions.

R. C. FAWDRY.

The President next called on Mr. A. Lodge for his Paper on :

AN ELEMENTARY METHOD OF FINDING CIRCLES OF CURVATURE AT POINTS, MULTIPLE OR OTHER, OF A PLANE CURVE WHOSE EQUATION IS GIVEN IN RECTANGULAR COORDINATES.

THE method of finding the centre of curvature to which I wish to draw attention is as follows :

If PT is the tangent at a point P of a curve, a circle which touches the curve at P will in general cut it at other points $Q, \ldots$.

Now, if P be taken as origin, and a homogeneous equation be formed by means of the equations of the curve and a tangent circle, this equation will represent all the chords $PQ, \ldots$. The circle of curvature will be distinguished by the fact that one of these chords must coincide with PT. That is the whole spirit of the method : there is no calculus, and the method is available for all who can transform an equation to a new point as origin and can appreciate the meaning of a homogeneous equation. In many cases, comprehension of what terms are small enough to be omitted will facilitate matters, but it is not essential, and indeed the method can be made to develop such comprehension.

Ex. 1. *To find the circle of curvature at* (0, 0) *of the curve*

$$(x+y)^2=a(x-y). \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Let its equation be

$$x^2+y^2=k(x-y). \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Then the homogeneous equation is

$$k(x+y)^2=a(x^2+y^2), \quad \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

and this will be satisfied by $x-y=0$ if

$$k=\tfrac{1}{2}a\,;$$

$\therefore$ the circle of curvature is

$$x^2+y^2=\frac{a}{2}(x-y), \quad \ldots\ldots\ldots\ldots\ldots\ldots (4)$$

and the centre of curvature is at $\left(\frac{a}{4}, -\frac{a}{4}\right)$.

Moreover, if in (3) we put $k=\frac{1}{2}a$, it becomes

$$(x+y)^2=2(x^2+y^2),$$

$$\text{i.e. } (x-y)^2=0.$$

Hence in this case the contact is of the 3rd order, as *all* the chords coincide with the tangent.

Ex. 2. *Case where the tangent at the origin is one of the axes.*

In this case the method leads to a quicker result.

Thus the circle of curvature at (0, 0) of $y^2=4ax$ is

$$x^2+y^2=4ax,$$

i.e. the presence or absence of x^2 in the presence of x has no effect on the curvature.

Similarly, in the case of $Ax^2+Bxy+Cy^2=x$, the circle of curvature at $(0, 0)$ is $Cx^2+Cy^2=x$.

Ex. 3. *To find the curvature at* $(at^2, 2at)$ *of the parabola* $y^2=4ax$.

We have $\quad (y-2at)^2=4a(x-ty+at^2)$,

or, taking P as origin, $\quad y^2=4a(x-ty)$.

Tangent circle: $\quad x^2+y^2=k(x-ty)$;

chord equation: $\quad ky^2=4a(x^2+y^2)$; $\therefore k=4a(1+t^2)$;

circle of curvature: $\quad x^2+y^2=4a(1+t^2)(x-ty)$;

revised chord equation: $\quad (1+t^2)y^2=x^2+y^2$,

$$\text{i.e. } (ty-x)(ty+x)=0;$$

showing that the remaining chord and the tangent are equally inclined to the axis.

The coordinates of the centre of curvature with the old origin are

$$(2a+3at^2, -2at^3).$$

The method is equally efficacious in the case of an ellipse or hyperbola.

Ex. 4. (*Double Point.*) *Find the curvature circle at* $(0, 0)$ *of the curve*

$$x(x^2+y^2)=a(x^2-y^2).$$

Take (1) $\quad x^2+y^2=k(x-y)$,

the chord being $\quad kx=a(x+y)$; $\therefore k=2a$;

(2) $\quad x^2+y^2=k(x+y)$,

the chord being $\quad kx=a(x-y)$; $\therefore k=2a$.

$\therefore$ the circles of curvature are

$$\text{(i) } x^2+y^2=2a(x-y); \quad \text{(ii) } x^2+y^2=2a(x+y).$$

For the curvature circle at $(a, 0)$, write $x+a$ for x, so as to move the origin to this point;

$$\therefore x(x^2+2ax+a^2)+y^2(x+2a)=0, \text{ i.e. } (x+2a)(x^2+y^2)+a^2x=0;$$

$\therefore 2y^2+ax=0$ is the parabola of curvature

and $\quad x^2+y^2+\frac{1}{2}ax=0$ is the circle of curvature.

The parabola of curvature to one of the branches at the origin can be obtained by the same method, but although the parabola has the interest of lying closer to the curve for a longer distance, it is not so easily drawn, nor is it quite so easily obtained.

However, taking $(Ax+By)^2=k(x-y)$ as a tangent parabola, we have

$$kx(x^2+y^2)=a(x+y)(Ax+By)^2$$

for chords;

$\therefore$ if $x=y$ is one of them, $\quad k=a(A+B)^2$;

$$\therefore (A+B)^2x(x^2+y^2)=(x+y)(Ax+By)^2.$$

To obtain a *second* chord coinciding with $x=y$ is rather lengthy unless we use the fact that if $f(y)=0$ has two equal roots, $f'(y)=0$ has the same root.

Using this method, we obtain

$$(A+B)^2 \cdot 2xy=(Ax+By)^2+2B(x+y)(Ax+By);$$

whence, if $x=y$, $\quad (A+B)^2=4B(A+B)$;

$$\therefore A+B=0$$

or $\quad A=3B.$

$A+B=0$ is useless, as it gives merely $(x-y)^2=0$.

$\therefore$ putting $A=3B$ in the equation of the parabola, we have

$$(3x+y)^2=16a(x-y)$$

as the equation of the parabola of curvature;

and the corresponding chord equation is

$$16x(x^2+y^2)=(x+y)(3x+y)^2,$$

which reduces to $$(x-y)^2(7x-y)=0.$$

$\therefore$ the curve cuts the parabola again on the line $y=7x$,

$$\text{i.e. where } (10x)^2=-96ax,$$
$$\text{i.e. } x=-0{\cdot}96a,$$
$$y=-6{\cdot}72a.$$

Other Examples.

For the curve $$(x^2+y^2)^2=a(3x^2y-y^3),$$

the circles of curvature at the origin are

$$x^2+y^2=3ay,$$
$$2(x^2+y^2)=3a(x\sqrt{3}-y),$$
$$2(x^2+y^2)+3a(x\sqrt{3}+y)=0;$$

the centres of curvature being equally spaced on the circumference of the circle, $4(x^2+y^2)=9a^2$.

For the curve $$y^3=(2x-3y)(3x+2y),$$

the origin circles are $$8(x^2+y^2)=169(2x-3y),$$
$$27(x^2+y^2)+169(3x+2y)=0.$$

For the curve $$y=e^x,\ (=1+x+\tfrac{1}{2}x^2+\ldots),$$

the curvature circle at (0, 1) is, when the origin is moved there,

$$x^2+y^2+4(x-y)=0.$$

For the curve $$x^2(x^2+y^2)=a^2(x-y)^2,$$

the curvature circles at the origin are

$$x^2+y^2=\pm a\sqrt{2}(x-y);$$

and when the origin is moved to $(-a, 0)$, the circle there is

$$3(x^2+y^2)=4a(x+y).$$

By help of these circles and the fact that $x=\pm a$ are asymptotes, the curve can be very accurately drawn with very little trouble.

The cases where $k=0$ or ∞ correspond to cusps and points of inflexion. In such cases the same method could be used to find semi-cubical or cubical parabolas which lie close to the curve. But this is an extension into which I do not propose now to go, though I have done it successfully in several cases.

A. Lodge.

GEOGRAPHICAL DISTRIBUTION OF MEMBERS.

A list of members of the Association on January 1st, 1915, classified according to locality, shows a total of 758, of whom England contains 565, Wales 31, Scotland 31, Ireland 13, Isle of Man 1, Channel Islands 2, South Africa 31, West Africa 2, Canada 6, Australia 22, New Zealand 6, West Indies 2, Malta 1, India 13, Straits Settlements 1, Egypt 5, Brazil 1,

China 1, Finland 1, Germany 2, Hungary 1, Italy 1, Japan 2, Persia 1, Roumania 1, Russia 2, Syria 1, U.S.A. 16.

The 565 members in England were distributed thus: London 149, Beds 10, Berks 11, Bucks 14, Cambs. 28, Cheshire 10, Cornwall 1, Cumberland 3, Derbyshire 6, Devon 11, Dorset 1, Durham 10, Essex 12, Glos 30, Hants 12, Herts 13, Hereford 3, Isle of Wight 3, Kent 25, Lancs 34, Lincs 1, Middlesex 7, Norfolk 6, Northants 3, Northumberland 5, Notts 6, Oxfordshire 13, Rutland 4, Shropshire 6, Somerset 7, Suffolk 3, Surrey 25, Sussex 16, Warwickshire 17, Westmorland 1, Wilts 2, Worcs 6, Yorkshire 41.

H. D. Ellis.

THE LONDON BRANCH.

The following papers were read at meetings of the London Branch during the summer and autumn of 1914:

"Simple Spherical Geometry." P. J. Harding.
"Pilot Balloons and the Determination of Wind Velocities." F. J. W. Whipple.
"The Teaching of Elementary Arithmetic." Mrs. F. G. Shinn.
"Some Points in very Elementary Algebra." L. W. Grenville.
"Puzzles." F. C. Boon.

The Branch President, Mr. F. W. Dyson, Astronomer Royal, exhibited and explained a machine for solving Spherical Triangles.

The meetings were well attended, and interesting discussions followed all the papers.

AUGUSTUS DE MORGAN.

The biography of De Morgan by his widow is now more than thirty years old, but the recent republication of three essays by him on Newton, reviewed in the October number of the *Gazette*, serves to recall one of the most striking figures in the London mathematical world of some fifty years ago. Augustus De Morgan created no new branches of knowledge and discovered little of note, yet when the scientific history of England in the nineteenth century is written his name will occupy a prominent position, for he profoundly influenced the opinions of the ordinary man of science and mathematician of his time.

De Morgan was born in 1806. In 1823, after a desultory education, he entered Trinity College, Cambridge, where he came under the influence of Peacock and Whewell. The time spent by a student at Oxford or Cambridge is frequently that in which his character and future course of life are determined, and, in my opinion, this was markedly so in De Morgan's case. His interests at the university spread far beyond the narrow limits of the Tripos, and philosophy, historico-mathematical subjects, music, and novels, in turn attracted his attention. Above all it was at Cambridge that he developed a rugged independence of character and a determination never to let his conduct be swayed by considerations of self-interest—life-long traits which explain various actions that struck his contemporaries as cranky. He had been brought up as a strict evangelical, but even as a lad felt unable to accept fully the beliefs of that school, and finally, though refusing to join any denomination, he came to sympathize generally with the unitarian attitude. Hence he declined to stand for a fellowship or proceed to the M.A. degree, both of which then involved a

declaration of belief; so to the loss of Cambridge, but gain of London, his direct connection with the former university ceased with his graduation in 1827.

In 1828 he was elected first professor of mathematics at what is now known as University College, London. His subsequent history was closely connected with it, and brings out clearly one side of his character. He resigned his chair in 1831 because the Council had used a power reserved to it of dismissing a teacher without assigning definite reasons. The regulation was subsequently altered, and on a vacancy occurring in 1836 he was reappointed professor. He contemplated resignation again in 1853 because the College accepted a legacy of books to be selected by members of the Church of England. In 1866 a graver issue was raised by the candidature of a unitarian minister for the chair of mental philosophy and logic. De Morgan held that the College was not entitled to consider the ecclesiastical position or creed of a candidate, and held that the refusal to appoint the particular candidate, who was otherwise excellently qualified, was really due to the fact that he held unpopular religious views. Accordingly he finally resigned his chair. His retirement was followed by family bereavements and by illness, and he only survived five years.

This bare sketch shows De Morgan as a man of high character, ever testing his conduct in the court of conscience, but it does not in any way explain the influence he exerted on his contemporaries. We cannot account for this influence by his professorial work, since his lectures, though stimulating, did not attract outside students of special ability; nor by his text-books, which are now nearly forgotten; nor by his investigations in formal logic, which, though excellent of their kind and paving the way to Boole's discoveries, appealed to but a limited class; nor by his connection with learned societies, which, though they brought him into contact with many men of science, gave him no exceptional facilities of intercourse; while we should have supposed that his authority would have suffered from his avowed belief in spiritualistic mediums, and his self-imposed rigid rules of conduct, which at different times led to his retirement from his professorship, his leaving the Council of the Astronomical Society, his refusal to allow himself to be nominated to a fellowship in the Royal Society, and his rejection of the offer of an honorary degree.

We are driven to seek elsewhere the secret of his undoubted power in the mathematical world, and I believe it is to be found in his historical papers and reviews, his occasional lectures on general subjects, and in the universal recognition of his desire for justice and scorn of all pretence. His bibliography of arithmetical books is a model of how such lists should be compiled, his essays on the calendar and almanacks are excellent specimens of historical research, and the success of the *Penny Cyclopaedia*, of which he wrote nearly one-sixth, was largely due to his articles. His *Budget of Paradoxes*, consisting of a reprint of some of the reviews he had written for the *Athenaeum*, shows humour as well as learning, and his papers, of which the mere list of titles occupies several pages of print in his memoir, cover a wide range of subjects and appealed to men of many tastes. In all this work he put himself in the most intimate relations with his readers, who must indeed have been unappreciative if they did not esteem the sincerity and learning of the writer. He was a good fighter, and in some of his letters admits that he loved controversy, but he was always scrupulously fair.

He wrote at length on extensions of formal logic, and had a sharp controversy with Sir William Hamilton on the subject. He himself attached great importance to these researches, but they now possess

little interest for the general reader. On the other hand, his scientific pursuits were many-sided, and in most of the discussions on subjects which interested him he took a leading part. He was a strong advocate of the introduction of decimal coinage, and studied the theory of probability especially in its application to life assurance; as I have indicated above, he defended the tenets of spiritualism; he also concerned himself with methods of scientific education, opposing all systems which involved competition; but in science, I think it was mathematico-historical questions, notably the follies of circle-squarers and the Newton-Leibnitz controversy, that specially attracted his attention, and by which he will hereafter be chiefly remembered.

The three *Essays on Newton* serve well to illustrate his historical research, love of justice, and vigorous style. The first essay was written for a collection of biographies which appeared in 1846 and contains an excellent account of Newton's life and writings: the value of his discoveries, his astonishing ability, and the high standard of his private life are fully recognized, but it is argued that he was of a morbidly suspicious temperament, and was unjust to Leibnitz and Flamsteed. It was well that these issues should be raised, for new and important evidence bearing on them had then been recently published. We may say that the charge in regard to Flamsteed is made out. The dispute between Newton and Leibnitz about the invention of the calculus stands on a somewhat different footing. In regard to it, De Morgan was an out-and-out supporter of Leibnitz, and put that side of the case forcibly. The second essay (1852) dealt entirely with this matter, as also did a considerable part of the third essay (1855), which is a review of Brewster's *Memoirs of Newton*. In these memoirs Brewster had gone at length into the controversy, and came to the opposite conclusion, namely, that Leibnitz was at fault. This verdict was warmly contested by De Morgan, who expressed surprise that the comments he had made on the subject in 1852 should have been ignored. I do not propose to discuss here this old and well-worn dispute. De Morgan showed that Leibnitz's case had not been fairly presented in 1712, but this does not settle the question at issue.

Much of De Morgan's *Budget of Paradoxes* is given up to circle-squarers. Its discursive character renders description almost impossible, but it brings out clearly his knowledge of books and men.

As illustrative of his ingenuity in applying the laws of probability to numerous problems, I will mention a test which he proposed for determining the authorship of books. He believed that if different books on similar subjects written by a particular author were examined it would be found that the average number of letters per word in each book would agree to (perhaps) one place of decimal. Hence, if the average number of letters per word in two books on the same subject differed by more than that percentage, it was probable that the books were by different authors. He thought this experiment might be well worth making in cases where authorship was in question, and in particular in the case of the Greek text of some of the books of the New Testament, but as far as I know the test has never been applied.

Although De Morgan declined to connect himself with any denomination, he accepted literally the historical statements in the New Testament, and was a convinced theist. But he held strongly that no one had a right to ask for information on his religious views, and never would allow a statement of his holding or rejecting any theological opinion to be used as a step towards position or material success. After his death it was found that he had inserted in his will a sentence to the effect that he had never confessed his belief that Christ was the Son of God, because during his life such confession had always been

the way up in the world. That refusal was characteristic of the man.

The emoluments of De Morgan's chair and his other earnings, though somewhat slight, were sufficient for his needs. The library which he gradually accumulated was, apart from review copies of books, mainly got by diligent search in second-hand shops and stalls, assuredly of all ways the most attractive. Fortunately, after his death it was purchased as a whole, and has been preserved; I do not think he would have wished a better memorial. Throughout his life he worked hard, and for several hours most evenings was accustomed to write articles and letters. It was difficult to tempt him from London, where alone he felt at home. It was never my good fortune to make his acquaintance, but when I was a small boy he was pointed out to me, walking down the Adelaide and Hampstead Roads from his residence to University College, touching with his forefinger every tenth post on his way, and I was told that to the residents on his route his daily progress was one of the incidents of the day. It was said that he knew certain posts which he would reach in due course, and if one of these did not come in the decennial series he was accustomed to go home and start again.

That De Morgan was obstinate and somewhat eccentric I readily admit, and I do not consider he was a genius, but he leaves on my mind the impression of a lovable man, with intense convictions, of marked originality, having many interests, and possessing exceptional powers of exposition. In those cases where his actions were criticized it would seem that the explanation is to be found in his determination always to take the highest standard of conduct without regard to consequences; he hated suggestions of compromise, expediency, or opportunism. Such men are rare, and we do well to honour them.

W. W. ROUSE BALL.

MATHEMATICAL NOTES.

436. [A. 1. a.] *On the sum of an* A.P.

The usual formula for the summation of an A.P.

$$S_n=\frac{n}{2}\{2a+(n-1)d\}$$

is a quadratic in n. To determine n, so as to satisfy given conditions for a, d and S (for S_n), write it

$$n^2d+n(2a-d)-2S=0. \qquad \text{(I.)}$$

1. Considering only those cases in which one root is a positive integer, *i.e.* in which there is at anyrate one solution with a concrete interpretation, we shall have a second root which must be rational, but which may be fractional or negative, or both.

Let the positive integral root be n_1 and the other n_2.

Then $$n_1+n_2=1-\frac{2a}{d} \text{ and } n_1n_2=-\frac{2S}{d},$$

and from these it is to be concluded that:

(α) n_2 will be positive if S and d have different signs; a and d must also have different signs.

(β) n_2 will be negative if S and d have the same sign.

(γ) n_2 will be integral if d is a factor of $2a$ and dn_1 is a factor of $2S$.

(δ) the denominator of a fractional solution will be d or a factor of d; it will be d if d is prime to $2a$.

2. n being a positive integer,

$$S_n \text{ is defined as } u_1+u_2+u_3+\ldots+u_n.$$

There are *a priori* reasons for expecting that S_{-n} might be numerically

$$u_{-1}+u_{-2}+u_{-3}+\ldots+u_{-n},$$
$$\text{or } u_1+u_0+u_{-1}+\ldots+u_{n-2},$$
$$\text{or } u_0+u_{-1}+u_{-2}+\ldots+u_{n-1}.$$

Consider n_2 to be $-m$ where m is integral.

Since this satisfies (I.), we have

$$S_{-m}=-\frac{m}{2}\{2a+(-m-1)d\}$$
$$=-\frac{m}{2}\{2(a-d)+(m-1)(-d)\} \quad \text{......................(II.)}$$
$$=-\{u_0+u_{-1}+u_{-2}+\ldots+u_{-m+1}\};$$

i.e. the summation of the series begins at the interval between u_0 and u_1, and includes n terms; these being counted backwards if n is negative and the sign of the sum being opposite to that of the arithmetical sum.

Under these circumstances, u_0 is a term in our sequence of terms; but S_0 is always 0, and may be regarded as the starting point of the summation, thus:

$$\begin{array}{ccccccccccccc} S_{-3} & & S_{-2} & & S_{-1} & & S_0 & & S_1 & & S_2 & & S_3 \\ & u_{-2} & & u_{-1} & & u_0 & \downarrow & u_1 & & u_2 & & u_3 & \end{array}$$

This arrangement does not give a symmetrical ordering for u and S at the same time and in the same way.

A particular case may be considered.

By formula, $n=4$ or -7 gives 28 as the sum of n terms of 4, 6, 8, ...

Writing the extended series thus:

$$\ldots, -10, -8, -6, -4, -2, 0, 2, \mid 4, 6, 8, 10, \ldots,$$

arithmetical addition confirms the general conclusion.

3. If n_2 is positive and fractional, it will be of the form $\frac{p}{q}$, where q is positive and numerically equal to d $\left[\frac{p}{q}\text{ is not reduced to its lowest terms}\right]$,

$$i.e. \quad q^2=d^2.$$

Then $$S_{\frac{p}{q}}=\frac{p}{2q}\left\{2a+\left(\frac{p}{q}-1\right)d\right\}, \text{ by substitution in (I.)},$$
$$=\frac{p}{2}\left\{2\left(\frac{a}{q}-\frac{q-1}{2}\cdot\frac{1}{d}\right)+(p-1)\frac{1}{d}\right\}; \quad \text{..............................(III.)}$$

i.e. $\frac{p}{q}$ can be interpreted by regarding $S_{\frac{p}{q}}$ as the sum of p terms of an auxiliary A.P. obtained by distributing each term of the original series into q terms, all the new terms being connected by the common difference $\frac{d}{q^2}$, *i.e.* $\frac{1}{d}$.

Again, taking a particular case: By formula, 3 or $3\frac{3}{4}$ terms of the series 11, 7, 3, ... add up to 21.

By (III.), the auxiliary series is obtained by distributing u_1 into four parts with a common difference of $-\frac{1}{4}$.

$$u_1=\tfrac{25}{8}+\tfrac{23}{8}+\tfrac{21}{8}+\tfrac{19}{8},$$
$$u_2=\tfrac{17}{8}+\tfrac{15}{8}+\tfrac{13}{8}+\tfrac{11}{8},$$
$$u_3=\tfrac{9}{8}+\tfrac{7}{8}+\tfrac{5}{8}+\tfrac{3}{8},$$
$$u_4=\tfrac{1}{8}+(-\tfrac{1}{8})+(-\tfrac{3}{8})+(-\tfrac{5}{8}), \text{ and so on.}$$

The sum of $3\frac{2}{4}$ terms then means either 14 terms of the auxiliary series or the sum of u_1, u_2, u_3 and the first two parts (according to this method of distribution) of u_4. The fractional value is thus interpreted.

The auxiliary series could also be written

$$\mid 6,\ 5,\ \mid 4,\ 3,\ \mid 2,\ 1,\ \mid 0,\ -1,\ \mid \ldots .$$

In this case the sum of $3\frac{1}{2}$ terms of the series = the sum of 3 terms, *i.e.* 21, "half the fourth term" (with this interpretation of a fraction of a term) being 0.

4. If n_2 is both fractional and negative, the results of §§ 3 and 4 must be combined, *i.e.* the series must be extended in the negative direction and the terms distributed to form an auxiliary A.P. as in § 3.

This attempt to interpret fractional and negative solutions for n in an A.P. arises naturally from a desire not to disregard the alternative root of the quadratic. This question would seldom arise for a G.P. (n would usually be either a positive integer or a logarithm), but analogy suggests that it be considered.

2B. If the real solution for n which satisfies $S=\dfrac{a(1-r^n)}{1-r}$ for given values of a, r and S is a negative integer, say $-m$, then

$$S_{-m}=\frac{a\left(1-\frac{1}{r^m}\right)}{1-r}=-\frac{\left(\frac{a}{r}\right)\left\{1-\left(\frac{1}{r}\right)^m\right\}}{1-\left(\frac{1}{r}\right)};$$

$$i.e.\quad S_{-m}=-\{u_0+u_{-1}+u_{-2}+\ldots+u_{-m+1}\}.$$

3B. If n is positive and fractional, say $\dfrac{p}{q}$,

$$S_{\frac{p}{q}}=\frac{a\left(1-r^{\frac{p}{q}}\right)}{1-r}=\left\{\frac{a\left(1-r^{\frac{1}{q}}\right)}{1-r}\right\}\left\{\frac{1-\left(r^{\frac{1}{q}}\right)^p}{1-r^{\frac{1}{q}}}\right\};$$

i.e. $S_{\frac{p}{q}}$ is the sum of p terms of an auxiliary G.P. obtained by distributing each term of the original G.P. into q parts so that the common ratio is $r^{\frac{1}{q}}$.

4B. For negative fractional values of n, the conclusions of 2B and 3B combined will hold.

The same methods of interpretation for negative and fractional values of n apply to both A.P. and G.P. But these interpretations for S_n do not give satisfactory results for u_n.

If, however, u_n be defined as u_0e^n, where e represents an operation performed on u_0 (*i.e.* in an A.P. as u_0+nd and in a G.P. as u_0r^n), and if S_n be defined as the result of making n additions of terms beginning with u_0,

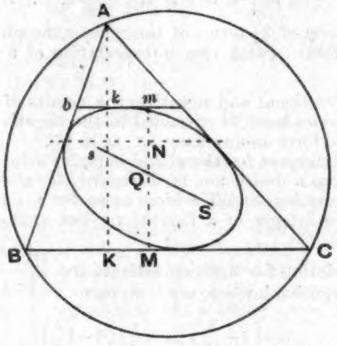
$$i.e.\quad S_n=u_0+u_1+u_2+\ldots+u_n,$$
$$S_{-n}=u_0+u_{-1}+u_{-2}+\ldots+u_{-n},$$

some better show of consistency in the behaviour of u_n and S_n can apparently be arranged. But the advantage is only apparent.

If other series, of which u_n is a rational integral algebraic function of n, such as n^2, $n(n+1)$, n^3, $n(n+1)(n+2)$, are considered, the interpretation of S_n for a negative integral value of n is the same as in A.P. and G.P.; for a fractional value of n the terms of the series must be distributed into terms of a series not of quite the same type, but of the same degree; but the method, that of differences, by which the auxiliary series is obtained, begs the issue.

F. C. BOON.

437. [K. 2. d. (vol. vi. p. 153, No. 349.)] Through Q, the centre of the inscribed ellipse, draw $Mm \parallel$ the $\perp^r$ AK to meet in m a tangent $bkm \parallel$ the base BC, and also to meet in N a line bisecting the sides. Then $KA = 2MN$; and as Q is midway between the tangents $Kk = 2MQ$, $\therefore$ $Ak = 2QN$.



Now the $\triangle^s$ Abs and ASC are similar (see No. 2 of the *Gazette*, or p. 311 of No. 73).

$$\therefore\ As \,.\, AS = Ab \,.\, AC = Ak \,.\, 2R \text{ from similar right-angled triangles}$$
$$= 4R \,.\, QN. \quad \text{Q.E.D.}$$

H. P. Rouse.

438. [J. 2. c.] *The St. Petersburg Problem.*

A man is to throw a coin until he throws head. If he throws head at the nth throw, *and not before*, he is to receive £2^n.

What is the value of his expectation? The paradox that the value of his expectation is infinite has caused an extensive literature, but I have not seen the following very obvious remark:

He is charged £1 for the (even) chance of getting £2 at his first throw, £1 for his chance of getting £4 at his second throw, and £1 for each possibility. Consequently it is not worth his while to pay more than n pounds to any person who could not pay him £2^n. The value of the chance is determined by the capital of the opponent.

C. S. Jackson.

439. [I. 5. a.] *The Definition of a Complex Number.*

It is generally recognised by now that a "complex number" $a + ib$ is merely a symbolic equivalent for a pair of real numbers (a, b).

Writers of elementary text-books have sometimes attempted to introduce complex numbers somewhat as follows.

Let $O(a)$ denote the result of an operation performed on a real number a, and such that

$$O\{O(a)\} = -a. \qquad \text{..........................(1)}$$

Then ia is identified with $O(a)$.

This procedure is open to a multitude of objections. For one thing, $O(a)$ is *ex hypothesi* an operation on a *real* number. If then $O(a)$ is not itself real, what is $O\{O(a)\}$? But there is an even more fatal objection, and the revival of the procedure in question in a recent book (a book too of some merit) leads me to think that it may be worth while to develop the objection in detail.

The objection is that the property (1) does not suffice to define any unique

operation. Let us suppose, for example, that O, operating on any complex number $a+ib$ or (a, b), produces

$$O(a, b)=(\lambda a+\mu b, \lambda' a+\mu' b),$$

where $\lambda, \mu, \lambda', \mu'$ are real.

Then
$$O\{O(a, b)\}=(A, B),$$
where
$$A=\lambda(\lambda a+\mu b)+\mu(\lambda' a+\mu' b),$$
$$B=\lambda'(\lambda a+\mu b)+\mu'(\lambda' a+\mu' b).$$

It will easily be verified that we shall have $A=-a$, $B=-b$, if the two conditions
$$\lambda+\mu'=0, \quad \lambda^2+\lambda'\mu=\lambda'\mu+\mu'^2=-1$$
are satisfied. We can satisfy these equations by taking

$$\lambda=\sinh\theta, \quad \mu=-\rho\cosh\theta, \quad \lambda'=\frac{\cosh\theta}{\rho}, \quad \mu'=-\sinh\theta,$$

θ and ρ being any real numbers; so that

$$O(a, b)=\left(a\sinh\theta-b\rho\cosh\theta, \ \frac{a\cosh\theta}{\rho}-b\sinh\theta\right).$$

In particular the operation O, when performed on a real number a, produces the complex number
$$a\sinh\theta+\frac{ia\cosh\theta}{\rho}.$$

The special case required is of course that in which $\theta=0$, $\rho=1$; when
$$O(a, b)=(-b, a), \quad O(a)=ia.$$
G. H. HARDY.

440. [K[1]. 7. a.] *To find the condition that the four straight lines represented by*
$$ax^4+4bx^3y+6cx^2y^2+4dxy^3+ey^4=0$$
shall form a harmonic pencil.

The lines joining the origin to the intersections of the straight lines $l_1x+m_1y-1=0$ and $l_2x+m_2y-1=0$ with the parabola $y^2=x$ have for their equation
$$\{y^2-x(l_1x+m_1y)\}.\{y^2-x(l_2x+m_2y)\}=0.$$

These lines form a harmonic pencil if $m_1m_2+2(l_1+l_2)=0$; that is to say, if the coefficient of y^2 in $(l_1x+m_1y-1)(l_2x+m_2y-1)=0$ is twice the coefficient of x.

The given equation may be written
$$ax^2+4bxy+2\lambda y^2+2(3c-\lambda)x(y^2/x)+4dy(y^2/x)+e(y^4/x^2)=0,$$
and therefore represents the straight lines joining the origin to the points of intersection of the parabola $y^2=x$ with the conic
$$ax^2+4bxy+2\lambda y^2+2(3c-\lambda)x+4dy+e=0.$$

This conic will represent two straight lines if
$$\begin{vmatrix} a & 2b & 3c-\lambda \\ 2b & 2\lambda & 2d \\ 3c-\lambda & 2d & e \end{vmatrix}=0,$$
and moreover, the joins of the origin to its intersections with $y^2=x$ will form a harmonic pencil if (as above) $2\lambda=12c-4\lambda$ or $\lambda=2c$.

Thus
$$\begin{vmatrix} a & 2b & c \\ 2b & 4c & 2d \\ c & 2d & e \end{vmatrix}=4\begin{vmatrix} a & b & c \\ b & c & d \\ c & d & e \end{vmatrix}=0,$$
which is the required condition. R. F. DAVIS.

441. [A.] (Vol. vii. p. 48, No. 391.) It may be worth noting that the symbol $\frown$, which is suggested by C. F. Boon (*Gazette*, No. 103, January 1913, p. 48) as an abbreviation for "approximately equal to," has been already so used. It occurs in Ernst Gottfried Fischer's *Lehrbuch der Elementar-Mathematik, vierter Theil:—Anfangsgründe der Algebra.* Berlin and Leipzig, 1829, p. 147 ff. R. C. ARCHIBALD.

442. [D. 6. b.] *Note on the Calculus for Non-Mathematicians.*

From Professor Love's book I learnt how to differentiate x^n without the aid of the binomial theorem, but the exponential limit when $\log x$ was reached proved troublesome. Now I find in Messrs. Barnard and Child's *Algebra* a proof—that $\left(1+\frac{1}{n}\right)^n$ increases as the positive integer n increases; thus, from Weierstrass's inequality,

$$\left(1-\frac{1}{nx}\right)^n > 1-\frac{1}{x}.$$

Thus

$$\left(1-\frac{1}{n^2}\right)^n > 1-\frac{1}{n};$$

so dividing by

$$\left(1-\frac{1}{n}\right)^n,$$

$$\left(1+\frac{1}{n}\right)^n > \left(1-\frac{1}{n}\right)^{-(n-1)}, \text{ i.e. } > \left(1+\frac{1}{n-1}\right)^{n-1}.$$

The binomial theorem is then used to establish the limit. We may, however, continue.

Hence

$$\left(1+\frac{1}{n}\right)^n < \left(1+\frac{1}{2n}\right)^{2n}$$

$$< \frac{1}{\left(1-\frac{1}{2n}\right)^{2n}} < \frac{1}{\left(1-\frac{1}{2}\right)^2},$$

or < 4,

since $\quad 1+\frac{1}{2n}$ is $< \left(1-\frac{1}{2n}\right)^{-1}$ and $\left(1-\frac{1}{2n}\right)^n$ is $> \left(1-\frac{1}{2}\right)$,

so that $\left(1+\frac{1}{n}\right)^n$ comes to a limit as the positive integer n increases.

In other positive values of n the expression can be trapped between

$$\left(1+\frac{1}{n}\right)^{n+1} \text{ and } \left(1+\frac{1}{n+1}\right)^n$$

in the usual manner, while for negative values we have

$$\frac{\left(1+\frac{1}{n}\right)^n}{\left(1-\frac{1}{n}\right)^{-n}} = \left(1-\frac{1}{n^2}\right)^n, \text{ which lies between 1 and } 1-\frac{1}{n}$$

when n is integral. Other negative values can be trapped like the positive ones.

C. H. Hardingham.

443. [L^2. 2. c.] It is a somewhat troublesome business to find the angle, θ, between the straight lines in which the plane

$$lx+my+nz=0 \quad \text{.................................(1)}$$

cuts the cone given by the general equation

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0. \quad \text{......................(2)}$$

The following is suggested as a neat determinant method in which the actual work is not heavy.

Let l, m, n in (1) be direction cosines; l_1, m_1, n_1; l_2, m_2, n_2, the direction cosines of the two lines in which (2) is cut by (1), so that

$$l_1l_2+m_1m_2+n_1n_2=\cos\theta.$$

Consider the product of

$$\Delta_1 \equiv \begin{vmatrix} 0 & l & m & n \\ l & a+\lambda & h & g \\ m & h & b+\lambda & f \\ n & g & f & c+\lambda \end{vmatrix} \text{ and } \Delta_2 \equiv \begin{vmatrix} 0 & l & m & n \\ 0 & l_1 & m_1 & n_1 \\ 0 & l_2 & m_2 & n_2 \\ 1 & 0 & 0 & 0 \end{vmatrix}.$$

Multiplying by rows, we have :

$$\Delta_1\times\Delta_2=\begin{vmatrix}1 & u+\lambda l & v+\lambda m & w+\lambda n\\ 0 & u_1+\lambda l_1 & v_1+\lambda m_1 & w_1+\lambda n_1\\ 0 & u_2+\lambda l_2 & v_2+\lambda m_2 & w_2+\lambda n_2\\ 0 & l & m & n\end{vmatrix},$$

where
$$u\equiv al+hm+gn,\quad u_1\equiv al_1+\dots,\quad u_2\equiv al_2+\dots,$$
$$v\equiv hl+bm+fn,\quad v_1\equiv hl_1+\dots,\quad v_2\equiv hl_2+\dots,$$
$$w\equiv gl+fm+cn,\quad w_1\equiv gl_1+\dots,\quad w_2\equiv gl_2+\dots,$$

since
$$l^2+m^2+n^2=1\,[=l_1^2+m_1^2+n_1^2=l_2^2+m_2^2+n_2^2];$$
$$ll_1+mm_1+nn_1=0=ll_2+mm_2+nn_2.$$

Multiplying again by Δ_2 by rows (taking the determinants as of the third order), we have :

$$-\Delta_1\times\Delta_2^2=\begin{vmatrix}P & Q & 1\\ \lambda & R+\lambda\cos\theta & 0\\ R+\lambda\cos\theta & \lambda & 0\end{vmatrix},$$

where
$$P\equiv l\,u_1+\dots\equiv l_1u\;+\dots\equiv all_1\;+\dots+f(m_1n_2+m_2n_1)+\dots,$$
$$Q\equiv l\,u_2+\dots\equiv l_2u\;+\dots\equiv all_2\;+\dots+\dots,$$
$$R\equiv l_1u_2+\dots\equiv l_2u_1+\dots\equiv al_1l_2+\dots+\dots,$$

since
$$l_1u_1+\dots\equiv al_1^2+\dots+2fm_1n_1+\dots=0,$$
$$l_2u_2+\dots\equiv al_2^2+\dots+2fm_2n_2+\dots=0.$$

Thus $\quad -\Delta_1\times\Delta_2^2\equiv\lambda^2-(R+\lambda\cos\theta)^2\equiv\lambda^2\sin^2\theta-2R\lambda\cos\theta-R^2,$

while
$$\Delta_2^2=\begin{vmatrix}1 & 0 & 0\\ 0 & 1 & \cos\theta\\ 0 & \cos\theta & 1\end{vmatrix}=\sin^2\theta.$$

If then we write $\qquad -\Delta_1\equiv\chi\lambda^2+\psi\lambda+\phi,$

we have, by equating coefficients of different powers of λ,

$$\chi=1 \text{ (as is easily verified)},$$
$$\psi\sin^2\theta=-2R\cos\theta,$$
$$\phi\sin^2\theta=-R^2.$$

Hence, eliminating R, we have $\tan^2\theta=\dfrac{-4\phi}{\psi^2}=\dfrac{-4\phi\chi}{\psi^2}.$

Since on expanding Δ_1 we have

$$\phi\equiv Al^2+\dots+2Fmn+\dots,$$
$$\psi\equiv(b+c)l^2+\dots-2fmn-\dots,$$
$$\chi\equiv l^2+m^2+n^2$$

(all homogeneous of the second degree in l, m, n), the final form for $\tan^2\theta$ will hold equally if l, m, n have any proportional values, and the restriction that they are direction cosines may be removed. PERCY J. HEAWOOD.

444. [A.1; B.1.] 1. Of the various methods of finding the condition that $u\equiv ax^2+by^2+cz^2+2fyz+2gzx+2hxy$ may break up into factors, some depend on algebraical manipulation which has no obvious relation to the determinant form of the result; others do not show clearly that the condition is both necessary and sufficient. The following is suggested as an interesting determinant method :

(i) *Unless u is a perfect square,*

$$by^2+2fyz+cz^2,\quad cz^2+2gzx+ax^2,\quad ax^2+2hxy+by^2$$

cannot all be perfect squares.

Suppose $ax^2+2hxy+by^2$ *not* a perfect square.

Then $$ax^2+2hxy+by^2 \equiv (\lambda_1 x+\mu_1 y)(\lambda_2 x+\mu_2 y), \qquad \text{(1)}$$
where $\begin{vmatrix} \lambda_1 & \mu_1 \\ \lambda_2 & \mu_2 \end{vmatrix}$ is *not* zero, ... (2)

Hence we have
$$u \equiv (\lambda_1 x+\mu_1 y+\nu_1 z)(\lambda_2 x+\mu_2 y+\nu_2 z)+\delta z^2, \qquad \text{(3)}$$
where ν_1, ν_2, δ are determined by
$$\lambda_2\nu_1+\lambda_1\nu_2=2g, \quad \mu_2\nu_1+\mu_1\nu_2=2f, \quad \nu_1\nu_2+\delta=c\,;$$
uniquely in consequence of (2).

By (3) we have $$\begin{vmatrix} \lambda_1 & \lambda_2 & 0 \\ \mu_1 & \mu_2 & 0 \\ \nu_1 & \nu_2 & 2\delta \end{vmatrix} \times \begin{vmatrix} \lambda_2 & \lambda_1 & 0 \\ \mu_2 & \mu_1 & 0 \\ \nu_2 & \nu_1 & 1 \end{vmatrix} \equiv \begin{vmatrix} 2a & 2h & 2g \\ 2h & 2b & 2f \\ 2g & 2f & 2c \end{vmatrix} = 8\Delta, \qquad \text{(4)}$$
multiplying by rows.

From (4), *in virtue of* (2), we see that $\delta=0$ is the necessary and sufficient condition that Δ may vanish. From (3), $\delta=0$ is the necessary and sufficient condition that u may break up into factors. Similarly whenever u is not a perfect square.

(ii) If u *is* a perfect square, and consequently the product of two factors, (3) may be satisfied with $\delta=0$ [though also in other ways]. Therefore (4) may be satisfied, and $\Delta=0$.

Therefore in all cases $\Delta=0$ is the necessary and sufficient condition that u may break up into two factors.

2. We may further proceed, in a somewhat similar manner, to discriminate in solid geometry between the different forms of surface defined by
$$u \equiv ax^2+by^2+cz^2+2fyz+2gzx+2hxy+2lx+2my+2nz+d=0,$$
in the cases where $D \equiv \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} =0$; and consequently, as just proved,
$$u \equiv LM+N \equiv (\lambda_1 x+\mu_1 y+\nu_1 z)(\lambda_2 x+\mu_2 y+\nu_2 z)+2lx+2my+2nz+d \quad \text{(say).}$$
From this identity it follows that
$$\begin{vmatrix} \lambda_1 & \lambda_2 & 2l & 0 \\ \mu_1 & \mu_2 & 2m & 0 \\ \nu_1 & \nu_2 & 2n & 0 \\ 0 & 0 & d & 1 \end{vmatrix} \times \begin{vmatrix} \lambda_2 & \lambda_1 & 0 & 2l \\ \mu_2 & \mu_1 & 0 & 2m \\ \nu_2 & \nu_1 & 0 & 2n \\ 0 & 0 & 1 & d \end{vmatrix} \equiv \begin{vmatrix} 2a & 2h & 2g & 2l \\ 2h & 2b & 2f & 2m \\ 2g & 2f & 2c & 2n \\ 2l & 2m & 2n & 2d \end{vmatrix} = 16\Delta,$$
multiplying by rows ;
$$i.e. \quad \Delta = \tfrac{1}{4} \begin{vmatrix} \lambda_1 & \lambda_2 & l \\ \mu_1 & \mu_2 & m \\ \nu_1 & \nu_2 & n \end{vmatrix}^2 .$$

This shows that $\Delta=0$ is the necessary and sufficient condition that the three planes $L=0$, $M=0$, $N=0$ may be perpendicular to the same plane; and taking such a plane as the plane of xy, $u=0$ reduces to
$$a'x^2+2h'xy+b'y^2+2l'x+2m'y+d'=0, \qquad \text{(1)}$$
showing that the surface is a cylinder.

If Δ is *not* zero, L, M, N cannot all be perpendicular to the same plane. Taking the xy plane perpendicular to L, M, the equation reduces to the form $a'x^2+2h'xy+b'y^2+2l'x+2m'y+2n'z+d'=0$; or with rotation of x, y axes,
$$\alpha x^2+\beta y^2+2\lambda x+2\mu y+2\nu z+\delta=0, \qquad \text{(2)}$$
where neither α, β *nor* ν *can be zero.** Therefore, with change of origin, we have $\alpha x^2+\beta y^2+2\nu z=0$ or $z=Ax^2+By^2$.

The usual way of proceeding, by observing that the Δ of (2) $=-\alpha\beta\nu^2$, and that therefore $\Delta=0$, *in the original equation*, is the condition that in (2)

* For if either were zero the planes L, M, N *would* be perpendicular to the same plane.

either a, β or ν must vanish, assumes that Δ is an invariant. There is, perhaps, no objection to this; yet it seems better in some ways to verify directly the expression of the original Δ in terms of the coefficients of L, M, N; and this leads almost at once to the result that the surface is a cylinder or paraboloid according as Δ is or is not zero. PERCY J. HEAWOOD.

THE PILLORY.

IN the Indian Civil Service examination, Aug. 1911, the following problem was set:

Prove that the series $\quad u_1+u_2+\dots+u_n+\dots,$

in which all the terms are positive, cannot be convergent unless nu_n tends to zero as a limit as n increases indefinitely.

This proposition is untrue, as the following series shows:

$$1+\tfrac{1}{2}+r^3+\tfrac{1}{4}+r^5+r^6+r^7+\tfrac{1}{8}+r^9+\dots,\quad 0<r<1,$$

where $\quad u_n=\dfrac{1}{2^s}$ if n is of the form 2^s $\quad$ and $\quad u_n=r^n$ otherwise.

This series converges to a value less than

$$(1+r+r^2+\dots)+(\tfrac{1}{2}+\tfrac{1}{4}+\tfrac{1}{8}+\dots)=\frac{1}{1-r}+1.$$

Furthermore, nu_n has no limit as n approaches infinity, but oscillates between 0 and 1. This disproves the proposition.

If it be required that the terms be *decreasing* as well as positive, or if it be required that nu_n shall approach a limit, the proposition holds.

Mathematical Department, Edinburgh University. LESTER R. FORD.

CORRESPONDENCE.

To the Editor of the Mathematical Gazette.

DEAR SIR,—Will you allow me a few lines in explanation of my criticism of Mr. Hatton's proof of Desargues' Theorem (*Math. Gazette*, pp. 394 and 435)?

I was wrong in my implication that Mr. Hatton had argued in a circle by assuming the Fundamental Theorem in his proof of Desargues' Theorem, and therefore withdraw with apologies the phrase "logical unsoundness" with its implied accusation. But my objection is by no means wholly removed.

Mr. Hatton's proof, while not assuming the entire "Fundamental Theorem," is based on the Perspective Proposition: "If $OLAA'$ and $OMBB'$ are two sets of collinear points such that each is in perspective with a third set $ONCC'$ (*i.e.* such that MN, BC, $B'C'$ are concurrent in a point X, and LN, AC, $A'C'$ in a point Y), then $OLAA'$ and $OMBB'$ are in perspective (*i.e.* LM, AB, $A'B'$ are concurrent in a point Z)." Employing only the Axioms of Connection (or Projective Axioms), Desargues' Theorem can be deduced from this, and conversely. There would therefore be no logical objection to a proof of Desargues' Theorem, using the Perspective Proposition as an additional axiom; but Mr. Hatton proceeds to prove the Perspective Proposition by using ratios of segments. I contend that there is a logical objection to this in a system of Pure Projective Geometry, to which segments and ratios of segments are foreign. Mr. Hatton is, of course, free to reply that his field of discourse, like that of most English treatises on "Projective Geometry"—Mr. G. B. Mathews' recent book is the only exception that I know of—is not Pure Projective Geometry but Metrical Geometry.

My objection then amounts to a protest against the use of extraneous subject matter in the proof of a theorem which is independent of such subject matter, just as I should object to the elementary proof of the existence of the common perpendicular to two skew lines on the ground that it employs the euclidean theory of parallels—always noting, however, that it is from the point of view of pure logic and not pedagogy that such objection is made.—Yours very truly,

D. M. Y. Sommerville.

13th January, 1915.

REVIEWS.

Elementary Applied Mechanics: Rules and Definitions. By W. G. Hibbins, B.Sc. Pp. 30. Price 6d. (Longmans.)

An exceedingly useful little pamphlet. A good feature is the care with which the language of the definitions is chosen. For instance, the moment of a force is defined as the "*tendency* which a force has to turn," and is intentionally and correctly divorced from the definition of the "measure of a moment,"—to which the attention of authors of text-books may be directed. The use of "velocity" and "linear velocity" for "speed" and "velocity" is hardly so happy: nor is the definition of momentum as a "property," especially as this definition is given several pages after Newton's second Law, in which the words "rate of change of momentum" are used.

A Senior Mental Arithmetic. By S. Gibson. Pp. 92. Price 1s. 6d. (Bell.)

Should prove useful to the boy or girl destined for trade. But from an educational point of view, the time would be much better spent in oral work in elementary algebraical substitutions and formula work in easy mensuration.

J. M. Child.

Baumé and Specific Gravity Tables. By N. H. Freeman. Pp. 27. Price 2s. 6d. 1914. (London: E. & F. Spon.)

Baumé's or Beaume's Hydrometer, extensively used in determining the alcoholic strength of spirit, is a variety of the common hydrometer, graduated by floating the instrument first in a ten per cent. solution of common salt, and then in pure distilled water at 54½° F. The interval between the water-lines is divided into 10 degrees.

Mr. Freeman has computed specific gravities to seven places of decimals with first and second differences to seven places, corresponding to Baumé degrees.

A labour of zeal and of love, but not according to knowledge. A specific gravity computed to seven significant figures, of which possibly three are reliable in favourable circumstances!

C. S. J.

Solutions of the Exercises in Godfrey and Siddons's Shorter Geometry. By E. A. Price, B.A. Pp. viii+160. 4s. 6d. net. 1914. (Cambridge University Press.)

Mr. Price's little volume will be found of considerable use to the private student, and even to teachers with such slender mathematical equipment that they are startled to find themselves in the presence of conics, conchoids, and the like (*e.g.* pp. 116-120). There are three pages of notes which will be useful to inexperienced teachers.

A Treatise on Differential Equations. By A. R. Forsyth, F.R.S. Pp. xviii+584. 14s. net. 1914. Fourth Edition. (Macmillan.)

The aim, scope, and signal merits of the above text-book are sufficiently well known to mathematicians in all countries to make a detailed notice of the fourth edition unnecessary. It will suffice to say that it is enlarged to the extent of some seventy additional pages. To quote from the preface, "Some portions of the book have been rewritten, particularly the early part of the chapter on the hypergeometric series, and parts of the chapter on

partial differential equations. A few additions have been made, each of them brief in itself. In association with the note on the method of Frobenius for the integrating of ordinary linear equations in series, I have given further notes on equations which have all or only some or even none of their integrals of the type called regular. The section dealing with total differential equations has been amplified so as to indicate the methods of obtaining an integral equivalent of Pfaffian equations when there are three variables and four variables respectively. In partial equations, the main changes are the insertion of a brief discussion of complete homogeneous linear systems of the first order, the use of these systems in the general construction of the intermediate integral of an equation of the second order when it possesses an intermediate integral, and a modified account of Ampère's method for equations of the second order."

Linear Algebras. By L. E. Dickson. Pp. viii+73. 3s. net. 1914. (Cambridge University Press.)

This is No. 16 of the *Cambridge Tracts in Mathematics and Mathematical Physics*. To those who have not the privilege of access to the *Encyclopaedia of Mathematics* and to British and foreign periodical literature of the science, this tract will be found a most useful compendium of matter introductory to "the general theory of linear algebras, including also non-associative algebras." After some thirty pages on definitions and elementary theorems, the reader finds a careful sketch of Cartan's general theory of complex linear associative algebras with a modulus. Part III. deals with the relations of linear associative algebras to linear groups and bilinear forms, and of linear algebras to finite groups. Part IV., on linear algebras over a field F, treats of the algebras of Weierstrass and division algebras, with a short note on analytic functions of hypercomplex numbers. Full reference to original sources is given throughout.

Problèmes d'Arithmétique Amusante. By P. Delens. Pp. viii+164. 2 fr. net. 1914. (Vuibert, Paris.)

This interesting little collection opens with a few exercises on the criteria of divisibility, followed by about 100 problems with solutions depending on very simple results in the theory of numbers. About 30 variations of the "think of a number type" are next dealt with and generalised, with extensions to more or less simple problems on cards, dice, and dominoes. The whole concludes with an interesting note on the rapid extraction of roots, given the cube of a number of four digits or the seventh power of a sum of three digits. There is no doubt nothing very novel about the problems to those who are familiar with the larger collections. But it is convenient for teachers to have such a selection in a cheap and handy form. More pupils than one would at first suspect can be led by easy trifles such as some of these are to experience a curiosity that must be further assuaged as to the laws of number. From personal experience, we can assert that not merely are such exercises in themselves amusing to the young, but in many and in sometimes unexpected instances a real taste for arithmetical research is inspired and set on the way to cultivation in earnest.

Algebraic Invariants. By L. E. Dickson. Pp. x+100. 5s. 6d. net. 1914. (Chapman & Hall.)

Prof. Dickson's monograph will prove a very useful introduction to the theory of invariants of algebraic forms. Part I. deals with geometrical interpretations of invariants and covariants. Part II. treats of the theory of invariants in non-symbolic notation. Part III. introduces the symbolic notation of Aronhold and Clebsch. The illustrative and other examples are carefully selected.

Les Coordonnées Intrinsèques. Théorie et Applications. By L. Braude. Pp. 100. 2 fr. 1914. (Gauthier-Villars.)

Cesàro's *Lezioni di geometria intrinseca* was published at Naples in 1895, and six years later was translated into German under the title *Vorlesungen über naturliche Geometrie* (Teubner, Leipzig). To avoid some of the disadvantages inherent in the use of cartesian and polar coordinates, it occurred

to K. F. Krause, early in the last century, to consider the equation of a plane curve in the form $f(s, \theta)=0$, (1) where s is the length of an arc P_0P of the curve measured from a fixed point P_0 to a variable point P, and θ is the angle between the tangents at P and P_0. The coordinates s, θ were called by A. Peters (1838) natural coordinates; Whewell, who had found them convenient in dealing with the cycloid and other curves, gave them the name of intrinsic coordinates. Casey and others wrote various memoirs on their applications, and Koestlin in recent years has studied the envelope of $x\cos\theta+y\sin\theta-s\cos\theta=0$, where s and θ are connected by the relation (1), and to this envelope he has given the name of the "arcuïde" of (1). The equation $\rho=\frac{ds}{d\theta}$ presented certain advantages in these and similar investigations, so a relation of the form $f(\rho, \theta)=0$ (2) became for a time the recognised intrinsic equation of a curve. It afforded a handy weapon with which to attack successive evolutes and the like, and was favoured in particular by Aoust in his work on the infinitesimal geometry of plane curves. With this ρ, θ as polar coordinates much was done by Col. Mannheim, and those who are familiar with the early volumes of the reprint from the *Educational Times* will remember certain papers by Mr. Tucker, who called (2) the "radial" of the primitive curve. But in later years the intrinsic equation $f(s, \rho)=0$ (3), with which Euler and others were in earlier days familiar, came into general use, and whatever credit is due for showing its value and exhibiting the elegance of its possibilities must be placed to the account of the gallant and ill-fated Italian, E. Cesàro. Treating s and ρ as cartesian rectangular coordinates, the curve (3) was studied by Aoust and Mannheim, and Wölffing has called it the "courbe de Mannheim" of the primitive curve. Sophus Lie pointed out that in the equations of forms $f\left(\rho, \frac{d\rho}{ds}\right)=0$, $f\left(\rho, \frac{d\rho}{d\theta}\right)=0$, the intrinsic coordinates are invariant for any translation or rotation in the plane.

Dr. Braude's little monograph in the "Scientia" series introduces the mathematical student to the subject, and by well-selected examples shows the value of the method and the simplicity of its applications in various fields. His first chapter sets forth general principles which are thus applied to special curves. The next is devoted to the Mannheim curve, *i.e.* the locus of the centre of curvature of the point of contact of a curve rolling on a fixed line or on a given curve. Then come the "arcuïdes," and, finally, we have numerous interesting applications to roulettes. Since the publication of this monograph, Dr. Braude has contributed a short paper to *L'Enseignement Mathématique* on a generalisation of the "arcuïde"—the curves $\rho=ae^{\sin\theta}\cos n\theta$, $\rho=a\theta e^{\sin\theta}$—logarithmoidal curves, causticoids of the logarithmic spiral.

Modern Instruments and Methods of Calculation. A Handbook of the Napier Centenary Exhibition. Edited by E. M. Horsburgh. Pp. vii + 343. 1914. (Bell & Sons.)

This is a handbook issued on the memorable occasion of Napier's tercentenary. Section A consists of a reprint of Prof. Gibson's sketch of the man and his work, from the *Proceedings of the Royal Philosophical Society of Glasgow.* Section B describes the loan collection of antiquarian relics, such as sets of bones, title-pages in facsimile of editions of Napier's works, portable sundials, photographs of the South Kensington calculating machines, and letters from mathematicians like Colin Maclaurin, Robert Simson, the Gregories, James Gregory, and others. Section C deals with Mathematical Tables, and here the reprint of Dr. Knott's account of Sang's "colossal work" will be new to many. Section D, on Calculating Machines, is prefaced by a general article from the pen of Mr. F. J. W. Whipple. Dr. Knott has disinterred from the *Transactions of the Asiatic Society of Japan* (1886) and abridged a most interesting article on the Abacus. Section F brings us to Slide Rules, which have a more obvious connection with logarithms than most of the exhibits. Section G consists of nearly 100 pages on instruments of calculation. Mathematical Models form the subject-matter of Section I, while K is devoted to Portraits and Medals. "An endeavour has been made to make the . . . Handbook useful to the laboratory computer, the engineer, the astronomer,

the statistician, and to all who are interested in calculation." The contributors have certainly gathered within the covers of the book a considerable body of information which otherwise would have to be gathered painfully from sources which to many people are practically inaccessible. It certainly emphasises the debt that is owed by the mathematical laboratory of this century to the invention of logarithms and to the marvels of mechanical ingenuity that eventually were to transform the labours of the calculators.

Memorabilia Mathematica, or The Philomath's Quotation-Book. By R. E. MORITZ. Pp. vii + 410. 12s. 6d. net. 1914. (Macmillan Co.)

There are few mathematicians who will not be grateful to Prof. Moritz for the collection of over 2000 notable passages which fill 383 pages of his *Memorabilia Mathematica*. For ten years he has been gathering passages dealing with mathematics and its various branches, and with mathematicians as men of science and as men of flesh and blood. One could fill pages of the *Gazette* with delightful samples of enthusiastic descriptions of our science, its influence on the individual mind, its value to the progress of mankind, the cause of its unpopularity, its relations to the other sciences, its aims, its permanency, and so forth. There is hardly a page on which is not to be found something to instruct or to titillate. Here and there, no doubt, the most learned among us will be struck by the abnormal depth of his own ignorance outside his special field—as, for instance, when Pierce gives the names of the first five poets in a biographical dictionary as Aagard, Abeille, Abulola, Abunowas, and Accords. The spasm of mingled astonishment and annoyance to which the five names may legitimately give rise will be at once dispelled for some by learning of the respective ages of these poets at their deaths, viz. 48, 76, 84, 48, 45; that: (*a*) the difference of the two digits 4 and 8, 7 and 6, etc., divided by 3 leaves *one*; (*b*) that $4^8 \div 3$, $7^6 \div 3$, etc., have remainders each *one*; (*c*) the sum of the prime factors of each age (including *one* as a prime factor) is a multiple of *three*. If we wonder how on earth Pierce found himself in this *galère*, we find that the passage is quoted from p. 163 of a work entitled "A Theory of Probable Inference." By this time one's senses are so rapidly dissipating that it is with furtive joy we note that the number of the page divided by *three* leaves a remainder *one*, and that the annoying 8's in 1883 add to 16, giving a remainder *one* on division by *three*, and thus leave us with a 1 and a 3. If we wonder what is the probable inference to be drawn from these alarming facts, the next page or so, p. 379 (also a number which divided by *three* leaves a *one*), reminds us that Pope was torn by a similar curiosity when in the *Dunciad* he makes Oldmixon in "naked majesty" exclaim:

"Ah! why, ye Gods, should two and two make four?"

It will be gathered from the above nonsense that the book is one which may be dipped into for amusement. But the reader can pull himself up in a moment if desired. We turn at random to p. 347 (also giving remainder ...), and we find nothing risible in the majestic solemnity of Schopenhauer's *Predicabilia a priori* (19 letters, also giving ...), or in a quotation of topical interest from the Platonic views as to the necessity of geometry to the soldier. Readers will be particularly grateful to Prof. Moritz for the catholicity of his selections, which range from Aeschylus to our new President, and for the many excellent quotations from familiar and unfamiliar American sources. He certainly appreciates to the full the flamboyant ecstasies of Sylvester and trenchant wit of De Morgan, for he seems to quote more from them than from any other writers. Not the least of the benefits he has conferred on us is the excellent Index of 27 pages.

Introduction Géométrique à quelques Théories Physiques. By E. BOREL. Pp. vii + 140. 5 frcs. 1914. (Gauthier-Villars.)

The first part of Prof. Borel's brochure consists of an elementary exposition of the theories of 4 and of *n*-dimensional geometry (*n* very large) as far as they serve for investigations in the theory of relativity and in statistical mechanics. The second part consists of seven papers reprinted from various sources and dealing with the following subjects: The principles of the Kinetic Theory of Gases; Statistical Mechanics and Irreversibility; Poincaré and the Relativity of Space; Some remarks on the Theory of Resonators; On a

Problem in Geometrical Probability; Kinematics in the Theory of Relativity; and Molecular Theories and Mathematics.

The author laments that in the French secondary instruction there is still far too great a gap between mathematics and reality. The aim of the mathematical discipline is to abstract from realities their common elements and to handle them so as to serve for the extension and creation of theories of as wide as possible a field of application. The child who knows that three trees with four apples a piece on them are bearing 12 apples, and yet is unable to utilise that fact in another domain, is far less advanced than the child who is not only aware that 3 times 4 are twelve, but can apply this result to simple problems. The essential characteristic of our science is this stripping from the formulae of experience their concrete content, and in their subsequent application to other and different concrete questions. Nowhere has this been found more fruitful than in the domain of Physics. The ball is being tossed backwards and forwards from the one science to the other. The mathematical theory of physical origin is developed in a purely abstract manner by the mathematician. It is used in its new form by the physicist and from it in time a new or extended mathematical theory emerges, and so the cycle of action and reaction goes on. But there may come a time when physical theories advance so rapidly that there has been hardly time enough for them to react upon Pure Mathematics. Such a time is the present, when we are more or less restricted to investigations of which the utility in physics is immediate. And with regard to these investigations, it must be remembered that their value persists long after the theories that inspired them, and provided them with material, have passed into the limbo that awaits so much of human speculation. Mathematical research on Laplace's equation is useful to the physicist long after he has abandoned Coulomb's electrostatics. The mathematical theory of periodical phenomena does not vanish with the mechanical theory and luminous phenomena which called it into existence. It is with this in mind, continues the author, that this little book is offered to the young mathematician, to induce him to interest himself in Mathematical Physics and those questions in Pure Mathematics which are connected with it. It is needless to add that, like most work from the pen of this brilliant mathematician and philosopher, this little volume is full of stimulus. Should one at times feel that he is giving one something on which "furieusement à penser," we at any rate are grateful for the conspicuous lucidity with which his argument is set out.

Lectures on the Icosahedron. By Felix Klein. Translated by Dr. G. G. Morrice. Pp. xvi + 289. 10s. 6d. net. Revised Edition. 1914. (Kegan, Paul & Co.)

Mathematicians with slenderly lined pockets—why not brush modesty aside, and say all mathematicians who have not this volume upon their shelves—will thank Dr. Morrice for bringing out this revised edition of a famous book. In the revision, the translator has had the advantage of the experience of Prof. Burnside, and it may be fairly said that we have now the definitive edition of the *Ikosaeder* in our tongue. The substance of the book appeared originally in papers contributed to the *Mathematische Annalen*, that publication with which the name of Klein has been so long and so honourably associated. It is a signal instance of the breadth of treatment so characteristic of the work of this distinguished mathematician, and marks a stage in his approach to the wide field of modular functions which he was at a subsequent period so thoroughly to explore. It would be interesting to have been honoured with the friendship of Klein in those young days when a bright future lay before him. His promise was recognised at an early date. Before he entered his twentieth year he was editing Plücker's posthumous work on Line Geometry. He had his doctorate as soon as this duty was discharged, and with Sophus Lie at Berlin he began what was to be a valued intimacy. At Berlin and in Paris they "jointly conceived the scheme of investigating geometric or analytic forms capable of transformation by means of groups of changes. This purpose has been of directing influence in our subsequent labours, though these may have appeared to lie far asunder. Whilst I primarily directed my attention to groups of discrete operations, and was thus led to the investigation

of regular solids and their relations to the theory of equations, Professor Lie attacked the more recondite theory of continued groups of transformations, and therewith of differential equations." But this period of fruitful interchange of views and of the inspiring play of mind on mind was rudely brought to a close. Paris was no place for the young German in the *année terrible*. He returned to his own country, and two years later he became Professor at Erlangen. In 1874 he was fortunate enough to come into "real contact" with Gordan, of the value of whose constant stimulus he speaks in glowing terms—he "has spurred me on when I flagged in my labours, and has helped me with the greatest disinterestedness over many difficulties which I should never have overcome alone."

The immediate outcome of this genial and persistent influence was the publication of the *Vorlesungen über das Ikosaeder und die Auflösung der Gleichungen vom fünften Grade*, 1884. The volume is divided into two parts —the theory of the icosahedron itself, and the theory of equations of the fifth degree. A discussion of the rotations of the regular bodies and the introduction of the conceptions of group-theory into the consideration of comparatively elementary geometrical figures is brought into relation first with the algebra of linear substitutions and the theory of invariants, and then with Riemann's theory of functions and the Galois theory of algebraical equations. We are shown how, by the repeated extractions of roots, the irrationalities in the equations of the dihedron, tetrahedron, and octahedron can be calculated, but that to deal with the icosahedron irrationality "an extension of the ordinary theory of equations seems to be indicated." When the icosahedral equation is solved, the question arises as to how far the results may be used in dealing with problems insoluble by mere extractions of roots. In the second part the leading conception of the theory of equations is presented in a geometrical form, equations of the fifth degree in which the third and fourth powers of the unknown are missing are discussed, the theory of the Jacobian equation of the sixth degree is revealed as open to attack by the help of the icosahedral equation, and equations of the third and fourth degree have fresh light thrown upon them by the handling of equations of the fifth degree. This sketch of the scope of the volume may turn the attention of younger readers to this famous little masterpiece. We might add that its relation to the general theory of groups of linear transformations, to the theory of substitutions, and to the great work on Modular Functions which appeared some six years after the *Ikosaeder*, is well sketched in a most interesting review of the *Elliptischen Modulfunctionen* written by Prof. F. N. Cole, who incidentally tells us much (that seems to be first hand) of the personality of the mathematician to whom the progress of the science in many fields is so deeply indebted. (*Bull. N.Y. Math. Soc.* 1891, Vol. I. pp. 105-120.)

Opere Matematiche di Luigi Cremona. In three volumes, published under the auspices of the R. Accademia dei Lincei. Vol. I. Pp. viii+497. 1914. (Hoepli, Milan.)

This stately tribute to the memory of the great Italian geometer will be welcomed by all who recognise the part that was played by Cremona in the development of the science to which he devoted his life. The thirty papers in the present volume date from 1855 to 1861. The most notable memoir here reprinted is the famous "Introduzione ad una teoria geometrica delle curve piane," published in the *Memorie dell' Accademia di Bologna* in 1862. As Enriques has said, this probably had more didactic influence than any other of his writings. With characteristic modesty the author offers it to the mathematical world as an incomplete sketch in which he has been inspired by the desire to find proofs of certain important theorems stated (and left, like Fermat's Theorems, without a hint of the method by which they were obtained) by Steiner in his short memoir entitled "Allgemeine Eigenschaften der algebraischen Curven." But it proceeds to deal with many other theorems, some original, and in particular with many obtained analytically by Plücker Salmon, Hesse, Cayley, Clebsch, and others. The special value of the memoir arises from the systematic plan of treatment, in which synthetic methods of wide scope reinforce the analytical methods of attack, and in which we see the opening movements of a campaign which was to leave a

wide field of investigation at the mercy of those who wielded the new weapons. The present volume is prefaced by a portrait of Cremona. To the remaining volumes we look forward with interest, not merely as possessions desirable in themselves, but with the hope that they may contain addresses, etc., hitherto out of the reach of English readers (such as the noble "Prolusione ad un Corso di Geometria Superiore," with its eloquent peroration beginning: "O giovani felici . . . ," p. 253), and a reasoned estimate of the work of the man, of its effect upon the Italian school of mathematicians, and of the final place that is to be awarded to him in the great company who felt the breath of mutual inspiration in an age of substantial and unprecedented developments.

Pure Mathematics. By G. H. HARDY. Second Edition. Pp. xii + 443. 12s. net. 1914. (Cambridge University Press.)

The extent of the principal changes made in this edition is conveniently stated in the preface by the author. In Chapter I. is inserted a sketch of Dedekind's theory of real numbers, and a proof of Weierstrass's theorem concerning points of condensation; in Chapter IV. an account of "limits of indetermination" and the "general principle of convergence"; in Chapter V. a proof of the Heine-Borel Theorem, Heine's theorem concerning uniform continuity, and Young's fundamental theorem concerning implicit functions; in Chapter VI. additional matter concerning the integration of algebraical functions. The sections which deal with the definition of the definite integral have been rewritten. One result, he adds, of all these alterations is to make the book more difficult, and we have the comforting assurance that "it is no longer necessary to apologise for treating mathematical analysis as a serious subject worthy of study for its own sake." Rearrangements and additions have brought the book from 419 to 442 pages.

THE LIBRARY.

THE Library has now a home in the rooms of the Teachers' Guild, 74 Gower Street, W.C. A catalogue has been issued to members containing the list of books, etc., belonging to the Association and the regulations under which they may be inspected or borrowed.

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BOOKS, ETC., RECEIVED.

Bolletino della "Mathesis." Dec. 1914. (Manuzio, Roma.)

Calculus Made Easy. By F. R. S. 2nd Edition, enlarged. Pp xii + 265. 2s. net. 1914. (Macmillan.)

Workshop Arithmetic. By F. CASTLE. Pp. viii + 172. 1s. 6d. 1915. (Macmillan.)

Plane Trigonometry. By H. S. CARSLAW. Second Edition. Pp. 293 + xi. 4s. *Solutions of the Questions.* Pp. 179. 6s. 6d. net. 1915. (Macmillan.)

Annuaire pour l'An 1915. Publié par le Bureau des Longitudes. Pp. 764 + 173. 1 fr. 50 c. 1914. (Gauthier-Villars.)

Exercises in Algebra (including Trigonometry). By T. P. NUNN. Part I., pp. x + 421. 4s.; without Answers, 3s. 6d. 1913. Part II., pp. xi + 551. 6s. 6d.; without Answers, 6s. 1914. (Longmans, Green.)

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